

Markov loops in discrete spaces

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1 Introduction

The main topic of these notes are Markov loops, studied in the context of continuous time Markov chains on discrete state spaces. We refer to [LJ11] and [Szn12] for the short "history" of the subject. In contrast with these references, symmetry is not assumed, and more attention is given to the infinite case. All results are presented in terms of the semigroup generator. In comparison with [LJ11], some delicate proofs are given in more details or with a better method. We focus mostly on properties of the (multi)occupation field but also included some results about loop clusters (see [LJL12] in the symmetric context) and spanning trees.

2 Preliminaries

In this section, we present some basic results about continuous time Markov chains, including a discrete version of Feynman-Kac formula and the transformation by time change.

2.1 Notations

1. Suppose E_1, E_2 are two countable sets, $(A_j^i, i \in E_1, j \in E_2)$ is a matrix. For $F_1 \subset E_1$ and $F_2 \subset E_2$, let $(A|_{F_1 \times F_2}, i \in F_1, j \in F_2)$ be the sub-matrix defined by $(A|_{F_1 \times F_2})_j^i = A_j^i$. By convention, the absolute value $|A|$ will denote the matrix: $(|A|)_j^i = |A_j^i|$.
2. $\mathcal{E}(\lambda), \lambda \in [0, \infty]$ denotes a random variable, exponentially distributed with parameter λ with the convention that $\mathcal{E}(0) = \infty$ and $\mathcal{E}(\infty) = 0$.
3. If k is a non-negative finite function on the state space S , M_k will denote the matrix, $(M_k)_y^x = k(x)\delta_y^x$ where $\delta_y^x = 1$ iff $x = y$.
4. $x \in \mathbb{R}^n$ can be extended to an periodic series, $x^{nm+k} = x^k, m \in \mathbb{Z}, k = 0, \dots, n-1$. Given $x \in \mathbb{R}^n$, each time we write x_{n+j} , we extend x to the n -periodical series.
5. For any countable set A , $\#A$ and $|A|$ will denote the number of elements in A .

6. Let \mathfrak{S}_k be the collection of permutations on $\{1, \dots, k\}$ and S some state space. For a permutation $\sigma \in \mathfrak{S}_k$ and $x = (x_1, \dots, x_k) \in S^k$, define $\sigma(x) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$. Accordingly, a permutation σ can be viewed as a function from S^k to S^k . Define the circular permutation r_j as follows: $r_j(1, \dots, k) = (j+1, \dots, k, 1, \dots, j)$. Define \mathfrak{R}_k to be the subset of \mathfrak{S}_k consisting of circular permutations on $\{1, \dots, k\}$. Note that σ plays two roles, a function on $\{1, \dots, k\}$ mapping an integer to another integer and a function on some S^k mapping a k -uple to another k -uple (for example, $r_1(2, 1, 3, 4) = (4, 2, 1, 3) \neq (r_1(2), r_1(3), r_1(1), r_1(4)) = (3, 4, 2, 1)$). We have $\sigma_1(\sigma_2(x_1, \dots, x_n)) = (x_{(\sigma_1 \circ \sigma_2)^{-1}(1)}, \dots, x_{(\sigma_1 \circ \sigma_2)^{-1}(n)})$.

2.2 Minimal continuous-time sub-Markov chain in a countable space

Let S be a countable set equipped with the discrete topology. Add an additional cemetery point ∂ to S and set $\bar{S} = S \cup \{\partial\}$ (compactification).

Definition 2.1 (Generator). A matrix $L = (L_y^x, x, y \in S)$ is called a sub-Markovian (Markovian resp.) generator iff

$$\begin{aligned} 0 &\leq -L_x^x < \infty && \text{for all } x \in S, \\ L_y^x &\geq 0 && \text{for all } x \neq y, \\ \sum_j L_y^x &\leq 0 \quad (\sum_j L_y^x = 0 \text{ resp.}) && \text{for all } x \in S. \end{aligned}$$

In case $L_x^x < 0$, set $Q_y^x = \frac{L_y^x}{-L_x^x}$ for $x \neq y$ and $Q_x^x = 0$. In case $L_x^x = 0$, set $Q_y^x = \delta_y^x$.

Convention 2.2. A sub-Markovian generator L on S can be extended to a Markovian generator \bar{L} on \bar{S} as follows: $\bar{L}_y^x = L_y^x$ for $x, y \in S$, $\bar{L}_\partial^x = -\sum_{y \in S} L_y^x$ for $x \in S$, $\bar{L}_x^\partial = 0$ for $x \in \bar{S}$.

Construction of the probability on the space of right continuous¹ paths

Let μ , a probability measure on S , be the initial distribution. Let ξ_0 be a random variable with distribution μ and $(\tau_{ix}, i \in \mathbb{N}, x \in S)$ be independent random variables, exponentially distributed with parameter $-L_x^x$. Let $(J_{ix}, i \in \mathbb{N}, x \in S)$ be independent random variables such that for $y \in S$

$$\mathbb{P}(J_{ix} = y) = Q_y^x.$$

¹In a discrete space, any right-continuous Markov chain has left limit in its lifetime $[0, \zeta[$ if the path stays at the cemetery ∂ after there has been infinitely many jumps. Besides, on $\zeta < \infty$, the left limit at time ζ is the cemetery point for the process.

Moreover, assume that $\xi_0, \tau = (\tau_{ix}, i \in \mathbb{N}, x \in S)$ and $J = (J_{ix}, i \in \mathbb{N}, x \in S)$ are independent. For any configuration of (μ, τ, J) , recursively define:

$$\begin{aligned}\xi_n &= J_{n\xi_{n-1}} \text{ for } n \geq 1 && \text{(discrete Markov chain)} \\ T_0 &= 0, T_{n+1} = T_n + \tau_{n\xi_n} && \text{(jumping time)} \\ \zeta &= \lim_{n \rightarrow \infty} T_n && \text{(explosion time).}\end{aligned}$$

Then define the path as follows:

$$\begin{aligned}X_t &= \xi_i \quad \text{for } T_i \leq t < T_{i+1}, \\ X_t &= \partial \quad \text{for } t \geq \zeta.\end{aligned}$$

Theorem 2.1 (Markov Property). *Set $(P_t)_y^x = \mathbb{P}[X_t = y | X_0 = x]$. Use \mathbb{P}^μ to stand for the law of the process $(X_t, t \geq 0)$. $(X_t, t \geq 0)$ defined above is a Markov process with initial distribution μ . Its semi-group will be denoted P_t and $(P_t)_y^x$ is right-continuous in t .*

The following theorem is taken from the book [Nor98].

Theorem 2.2.

a) **Backward Equation.**

P_t is the minimal non-negative solution of the backward equation:

$$\begin{aligned}\frac{dP_t}{dt} &= LP_t \\ P_0 &= I \text{ (identity).}\end{aligned}$$

b) **Forward Equation.**

P_t is the minimal non-negative solution of the forward equation:

$$\begin{aligned}\frac{dP_t}{dt} &= P_t L \\ P_0 &= I \text{ (identity).}\end{aligned}$$

(These equations are viewed as an infinite system of differential equations.)

Remark 1. The process we constructed is minimal in the sense of its semi-group as the solution of the forward backward equations. In a more probabilistic language, it is the least conservative process. To be more precise, for any sub-Markovian process with generator L , if we kill the process as long as it jumps infinitely many times, we get the minimal sub-Markov process with generator L .

Definition 2.3. The potential V is defined as follows:

$$V_y^x = \mathbb{E}^x \left[\int_0^\infty 1_{\{X_t=y\}} dt \right] = \int_0^\infty (P_t)_y^x dt.$$

Theorem 2.3 (Feynman-Kac). *For a non-negative function k on S , define*

$$(P_{t,k})_y^x = \mathbb{E}^x \left(e^{-\int_0^t k(X_s) ds} 1_{\{X_t=y\}} \right).$$

Then, it is the minimal positive solution of the following equation:

$$\frac{\partial u}{\partial t}(t, x) = (L - M_k)u(t, x) \quad ^2$$

with initial condition $u(0, x) = \delta_y^x$. We denote by V_k the associated potential. Denote by \mathbb{P}_k the law of the canonical minimal Markov process with generator $L - M_k$. Then,

$$\frac{d\mathbb{P}_k}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{-\int_0^t k(X_s) ds}$$

where $\mathcal{F}_t = \sigma(X_s, s \in [0, t])$.

Proposition 2.4. *Suppose V is transient, i.e. $V_y^x < \infty$ for all x and y , then $LV = VL = -Id$.*

Theorem 2.5 (Resolvent equation). *The following identities hold:*

- a) $V_k + VM_kV_k = V$.
- b) $V_k + V_kM_kV = V$.
- c) $V_kM_kV = VM_kV_k$.

2.3 The time change induced by a non-negative function

Let $(X_t, t \geq 0)$ be a minimal Markovian process on S , with generator L and lifetime ζ . Given $\lambda : S \rightarrow [0, \infty]$, define

$$A_t = \int_0^{t \wedge \zeta} \lambda(X_s) ds, \quad \sigma_t = \inf\{s \geq 0, A_s > t\}, \quad \hat{\zeta} = \inf\{s \geq 0, \sigma_s = \sigma_\infty\}$$

with the convention that $\inf \emptyset = \infty$. Then, σ_t are stopping times for all t and they are right-continuous with respect to $t \geq 0$. Set $Y_t = X_{\sigma_t}$ for $0 \leq t \leq \hat{\zeta}$ and let Y be killed at time $\hat{\zeta}$. By the strong Markov property, Y_t is also a càdlàg sub-Markov process with lifetime $\hat{\zeta}$. It could be constructed directly from its generator \hat{L} as before.

Proposition 2.6.

²Recall that $(M_k f)(x) = k(x)f(x)$.

a) If $0 < \lambda < \infty$, then $\hat{L}_y^x = \frac{L_y^x}{\lambda_x}$ (change of jumping rates).

b) If $\lambda = 1_A + 1_{A^c} \cdot \infty$, then

$$\hat{L}_y^x = \begin{cases} L_y^x & \text{for } x, y \in A^c \\ 0 & \text{elsewhere.} \end{cases}$$

(Y is the restriction of X to A .)

c) If $\lambda = 1_A$, Y is called the trace of X on A . The generator \hat{L} of Y will be denoted by L_A . In this case, $(Y_t, t \geq 0)$ has the same potential as $(X_t, t \geq 0)$. On $A \times A$:

$$V_y^x = \mathbb{E}^x \left[\int_0^\zeta 1_{\{X_s=y\}} ds \right] = \mathbb{E}^x \left[\int_0^{\hat{\zeta}} 1_{\{Y_s=y\}} ds \right].$$

Let T_1 be the first jumping time and $T_{1,A} = \inf\{s \geq T_1, X_s \in A\}$.

Define $(R^A)_y^x = \mathbb{E}^x[X_{T_{1,A}} = y, T_{1,A} < \infty]$ for $y \in S$ and $(R^A)_\partial^x = 1 - \sum_y (R^A)_y^x$. Then, the generator L_A of Y satisfies:

$$(L_A)_x^x = L_x^x(1 - (R^A)_x^x) \text{ and } (L_A)_y^x = -L_x^x(R^A)_y^x \text{ for } x \neq y.$$

Proof. Define $T_A = \inf\{t \geq 0, X_t \in A\}$ and $(H_A)_y^x = \mathbb{E}^x[X_{T_A} = y, T_A < \infty]$. As usual, set $Q_y^x = -\frac{L_y^x}{L_x^x}$ for $y \neq x$, $Q_x^x = 0$ and $Q_y^x = \delta_y^x$ if $L_x^x = 0$. For any subset B of S , define $(J_B)_y^x = 1_{\{x \in B\}} \delta_y^x$, $G_B = I + QJ_B + QJ_BQJ_B + \dots$. Then

$$H_A = J_A + J_{A^c}QH_A = J_A + J_{A^c}QJ_A + J_{A^c}QJ_{A^c}QJ_A + \dots$$

Next, we see that $(R^A)_y^x = \mathbb{E}^x[X_{T_{1,A}} = y] = Q_y^x + \sum_{z \in A^c} Q_z^x(H_A)_y^z = (G_{A^c}QJ_A)_y^x$ for $x, y \in A$. Then, Y can be described as follows: from x , it waits for an $\mathcal{E}(-L_x^x)$ -time, then jumps to $y \in A \cup \{\partial\}$ according to $(R^A)_y^x$ (it does not actually jump if $y = x$). Finally, it is not hard to see that $(L_A)_x^x = L_x^x(1 - (R^A)_x^x)$ and $(L_A)_y^x = -(R^A)_y^x L_x^x$ for $y \neq x$. \square

Definition 2.4. For $A \subset S$, define $V_A = V|_{A \times A}$. V_A is the potential of the trace of the Markov process on A and $L_A = -(V_A)^{-1}$ is its generator. Let $L^A = L|_{A \times A}$ denote the generator of the Markov process restricted in A (i.e. killed at entering A^c) and let $V^A = (-L^A)^{-1}$ be its potential.

Proposition 2.7. Assume that V is transient, χ is a non-negative function on S and that $F \subset S$ contains the support of χ . Then, $(V_\chi)_F = (V_F)_\chi$.

3 Loops and Markovian loop measure

In this section, we introduce the loop measure associated with a continuous time Markov chain. Its properties under various transformations (time change, trace, restriction, Feynman-Kac) are studied as well as the associated occupation and multi-occupation field.

3.1 Definitions and basic properties

Definition 3.1 (Based loops). A based loop l is an element $(\xi_1, \tau_1, \dots, \xi_p, \tau_p, \xi_{p+1}, \tau_{p+1})$ in $\bigcup_{p \in \mathbb{N}} (S \times]0, +\infty[)^{p+1}$ such that $\xi_{p+1} = \xi_1$ and $\xi_{i+1} \neq \xi_i$ for $i = 1, \dots, p$. We call p the number of jumps in l and denote it by $p(l)$. Define $T = \tau_1 + \dots + \tau_{p+1}$, $T_0 = 0$, $T_i = \tau_1 + \dots + \tau_i$. Then, a based loop can be viewed as a càdlàg piecewise constant path l on $[0, T]$ such that $l(t) = \xi_{i+1}$ for $t \in [T_i, T_{i+1}[$, $i = 1, \dots, p$ and $l(T) = \xi_{p+1} = \xi_1$. Clearly, we have $l(T) = l(T-)$.

Let \mathbb{P}^x be the law of the minimal sub-Markovian process started from x with semi-group $(P_t, t \geq 0)$ (or with generator L equivalently). It induces a probability measure on the space of paths l indexed by $[0, t]$, namely \mathbb{P}_t^x . \mathbb{P}_t^x is carried by the space of paths with finite many jumps such that $l(0) = l(0+) = x$. Define the non-normalized bridge measure $\mathbb{P}_{t,y}^x$ from x to y with duration time t as follows: $\mathbb{P}_{t,y}^x(\cdot) = \mathbb{P}_t^x(\cdot \cap 1_{\{l(t)=y\}})$.

Definition 3.2. The measure on the based loops is defined as $\mu^b = \sum_{x \in S} \int_0^\infty \frac{1}{t} \mathbb{P}_{t,x}^x dt$.

Proposition 3.1 (Expression of the based loop measure). *For $k \geq 2$,*

$$\begin{aligned} \mu^b(p(l) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \xi_{k+1} = x_{k+1}, \tau_1 \in dt^1, \dots, \tau_{k+1} \in dt^{k+1}) \\ = 1_{\{x_1 = x_{k+1}\}} Q_{x_2}^{x_1} \dots Q_{x_1}^{x_k} \frac{1}{t^1 + \dots + t^{k+1}} (-L_{x_1}^{x_1}) e^{L_{x_1}^{x_1} t^1} \dots (-L_{x_k}^{x_k}) e^{L_{x_k}^{x_k} t^k} e^{L_{x_{k+1}}^{x_{k+1}} t^{k+1}} dt^1 \dots dt^{k+1} \end{aligned}$$

For $k = 1$,

$$\mu^b(p(l) = 1, \xi = x, \tau \in dt) = \frac{1}{t} e^{L_x^x t} dt$$

Proof. For $k \geq 2$ and all sequence of positive measurable functions $(f_i, i \geq 1)$, denote by $(*)$ the value of $\mu^b(p(l) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \xi_{k+1} = x_{k+1}, f_1(\tau_1) \dots f_{k+1}(\tau_{k+1}))$.

$$\begin{aligned} (*) &= \int_0^\infty \frac{dt}{t} \sum_{x \in S} \mathbb{P}_t^x[p(l) = k, \xi_1 = x_1, \dots, \xi_{k+1} = x_{k+1}, f_1(\tau_1) \dots f_k(\tau_k) f_{k+1}(t - \sum_{j=1}^k \tau_j)] \\ &= \int_0^\infty \frac{dt}{t} \sum_{x \in S} \mathbb{P}_t^x[p(l) = k, \xi_1 = x_1, \dots, \xi_{k+1} = x_{k+1}, \\ &\quad f_1(\tau_1) \dots f_k(\tau_k) f_{k+1}(t - \sum_{j=1}^k \tau_j), l(t) = x] \end{aligned}$$

$$= \int_0^\infty \frac{dt}{t} \mathbb{P}_t^{x_1} [p(l) = k, \xi_1 = x_1, \dots, \xi_{k+1} = x_{k+1}, \\ f_1(\tau_1) \cdots f_k(\tau_k) f_{k+1}(t - \sum_{j=1}^k \tau_j), l(t) = x_1].$$

By definition of \mathbb{P}_t^x ,

$$(*) = 1_{\{x_1=x_{k+1}\}} \int_0^\infty \frac{1}{t} dt Q_{x_2}^{x_1} \cdots Q_{x_k}^{x_{k-1}} Q_{x_1}^{x_k} \int_{\substack{s^1, \dots, s^{k+1} > 0 \\ s^1 + \dots + s^k < t \\ s^1 + \dots + s^{k+1} > t}} f_1(s^1) \cdots f_k(s^k) f_{k+1}(t - \sum_{j=1}^k s_j) \\ \left(\prod_{i=1}^{k+1} (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} s^i} ds^i \right) \\ = 1_{\{x_1=x_{k+1}\}} \int_0^\infty \frac{dt}{t} Q_{x_2}^{x_1} \cdots Q_{x_k}^{x_{k-1}} Q_{x_1}^{x_k} \int_{\substack{s^1, \dots, s^k > 0 \\ s^1 + \dots + s^k < t}} f_1(s^1) \cdots f_k(s^k) f_{k+1}(t - s^1 - \dots - s^k) \\ e^{L_{x_1}^{x_1}(t-s^1-\dots-s^k)} \left(\prod_{i=1}^k (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} s^i} ds^i \right).$$

Now, change the variables as follows: $t^1 = s^1, \dots, t^k = s^k, t^{k+1} = t - s^1 - \dots - s^k$.

$$\mu^b(p(l) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \xi_{k+1} = x_{k+1}, f_1(\tau_1) \cdots f_k(\tau_k) f_{k+1}(\tau_{k+1})) \\ = 1_{\{x_1=x_{k+1}\}} \int_{\substack{t^1, \dots, t^{k+1} > 0}} \frac{1}{t^1 + \dots + t^{k+1}} Q_{x_2}^{x_1} \cdots Q_{x_k}^{x_{k-1}} Q_{x_1}^{x_k} f_1(t^1) \cdots f_k(t^k) f_{k+1}(t^{k+1}) e^{L_{x_1}^{x_1} t^{k+1}} \\ \prod_{i=1}^k (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} t^i} \prod_{i=1}^{k+1} dt^i.$$

Consequently, for $k \geq 2$,

$$\mu^b(p(l) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \xi_{k+1} = x_{k+1}, \tau_1 \in dt^1, \dots, \tau_{k+1} \in dt^{k+1}) \\ = 1_{\{x_1=x_{k+1}\}} Q_{x_2}^{x_1} \cdots Q_{x_k}^{x_{k-1}} \frac{1}{t^1 + \dots + t^{k+1}} (-L_{x_1}^{x_1}) e^{L_{x_1}^{x_1} t^1} \cdots (-L_{x_k}^{x_k}) e^{L_{x_k}^{x_k} t^k} e^{L_{x_{k+1}}^{x_{k+1}} t^{k+1}} dt^1 \cdots dt^{k+1}.$$

The case $k = 1$ is similar and even simpler. \square

Definition 3.3 (Doob's harmonic transform). A non-negative function h is said to be excessive iff $-Lh \geq 0$. Define Doob's harmonic transform $((L^h)_y^x, x, y \in \text{supp}(h))$ of L as follows

$$(L^h)_y^x = \frac{L_y^x h(y)}{h(x)}.$$

As in [LJL12], the following proposition is a direct consequence of Proposition 3.1.

Proposition 3.2. *The based loop measure is invariant under the harmonic transform with respect to any strictly positive excessive function.*

Remark 2. Doob's h -transform with respect to a strictly positive function does not change the bridge measure.

Definition 3.4 (Pointed loops and discrete pointed loops). Using the same notation as before, set $\tau_1^* = \tau_1 + \tau_{p(l)+1}$, $\tau_i^* = \tau_i$ for $1 < i < p(l) + 1$. Then $(\xi_1, \tau_1^*, \dots, \xi_{p(l)}, \tau_{p(l)}^*) \in \bigcup_{p \in \mathbb{N}_+} (S \times \mathbb{R}_+)^p$ is called the pointed loop obtained from the based loop $(\xi_1, \tau_1, \dots, \xi_{p(l)+1} = \xi_1, \tau_{p(l)+1})$. Clearly, $\xi_1 \neq \xi_{p(l)}$ and $\xi_i \neq \xi_{i+1}$ for $i = 1, \dots, p-1$. The induced measure on pointed loops is denoted by μ^p . By removing the holding times from the pointed loop, we get a discrete based loop $\xi = (\xi_1, \dots, \xi_{p(l)})$.

As a direct consequence of Proposition 3.1, we obtain the following by change of variables:

Proposition 3.3 (Expression of μ^p). *For $k \geq 2$,*

$$\begin{aligned} \mu^p(p(l) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \tau_1^* \in dt^1, \dots, \tau_k^* \in dt^k) \\ = Q_{x_2}^{x_1} \dots Q_{x_1}^{x_k} \frac{t^1}{t^1 + \dots + t^k} (-L_{x_1}^{x_1}) e^{L_{x_1}^{x_1} t^1} \dots (-L_{x_k}^{x_k}) e^{L_{x_k}^{x_k} t^k} dt^1 \dots dt^k. \end{aligned}$$

For $k = 1$,

$$\mu^p(p(l) = 1, \xi_1 = x_1, \tau_1^* \in dt^1) = \frac{1}{t^1} e^{L_{x_1}^{x_1} t^1} dt^1.$$

Definition 3.5 (Loops and loop measure). We define an equivalence relation between based loops. Two based loops are called equivalent iff they have the same time length and their periodical extensions are the same under a translation on \mathbb{R} . The equivalence class of a based loop l is called a loop and denoted l^o . Sometimes, for the simplicity of the notations, if there is no ambiguity, we will omit the superscript o and use the same notation l for a based loop and the associated loop. Moreover, the based loop measure induces a measure on loops, namely the loop measure μ . The loop measure is defined by the generator L . Sometimes, we will write $\mu(L, dl)$ instead of μ to stress this point.

Definition 3.6. For a pointed loop l , let $p(l)$ be the number of jumps made by l . For any pointed loop $(\xi_1, \tau_1, \dots, \xi_n, \tau_n)$, define $N_y^x = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x, \xi_{i+1}=y\}}$ and $N^x = \sum_{y \in S} N_y^x = \sum_{i=1}^{p(l)} 1_{\{\xi_i=x\}}$. $p(l)$, $N_y^x(l)$ and $N^x(l)$ have the same value for equivalent pointed loops. Accordingly, they can be defined on the space of loops and denoted the same.

Definition 3.7 (Discrete loops and discrete loop measure). We define an equivalence relation \sim on $\bigcup_k S^k$ as follows: $(x_1, \dots, x_n) \sim (y_1, \dots, y_m)$ iff $m = n$ and $\exists j \in \mathbb{Z}$ such that $(x_1, \dots, x_n) = (y_{1+j}, \dots, y_{m+j})$. For any $(x_1, \dots, x_n) \in \bigcup_k S^k$, use $(x_1, \dots, x_n)^o$ to stand for the equivalent class of (x_1, \dots, x_n) . Then the space of discrete loops is $\{(x_1, \dots, x_n)^o; (x_1, \dots, x_n) \in \bigcup_k S^k\}$. For any loop $l^o = (x_1, t^1, \dots, x_k, t^k)^o$, use $l^{o,d}$ to

stand for the discrete loop $(x_1, \dots, x_k)^o$. The mapping from loops to discrete loops and the loop measure induces a measure on the space of discrete loops, namely the discrete loop measure μ^d .

Definition 3.8 (Powers). Let $l : [0, |l|] \rightarrow S$ be a based loop. Define the n -th power of $l^n : [0, n|l|] \rightarrow S$ as follows: for $k = 0, \dots, n-1$ and $t \in [0, |l|]$, $l^n(t+k|l|) = l(t)$. The n -th powers of equivalent based loops are again equivalent. Consequently, the n -th powers of the loop is well-defined. The powers of the discrete loops are defined similarly.

Definition 3.9 (Multiplicity and primitive of the non-trivial loops). The multiplicity of a discrete loop is defined as follows:

$$n(l^{o,d}) = \max\{k \in \mathbb{N} : \exists \tilde{l}^{o,d}, l^{o,d} = (\tilde{l}^{o,d})^k\}$$

If $l^{o,d} = (\tilde{l}^{o,d})^{n(l^{o,d})}$, then $\tilde{l}^{o,d}$ is called a primitive of $l^{o,d}$. For a non-trivial loop l , the multiplicity is defined as follows:

$$n(l^o) = \max\{k \in \mathbb{N} : \exists \tilde{l}^o, l^o = (\tilde{l}^o)^k\}$$

For a trivial loop l , the multiplicity is defined to be 1. If $(\tilde{l}^o)^{n(l^o)} = l^o$, then \tilde{l}^o will be called the primitive of l^o , as it is always unique. And we will use *prime* to stand for the mapping from a (discrete) loop to its primitive.

Definition 3.10 (Primitive (discrete) loops and (discrete) primitive loop measure). A (discrete) loop is called primitive iff its multiplicity is one. The mapping *prime* induces a measure on (discrete) primitive loops, namely the (discrete) primitive loop measure.

Proposition 3.4. *We have the following expression for the discrete loop measure:*

$$\mu^d((x_1, \dots, x_k)^o) = \frac{1}{n((x_1, \dots, x_k)^o)} Q_{x_2}^{x_1} \dots Q_{x_1}^{x_k}.$$

Definition 3.11 (Pointed loop measure). We can define another measure μ^{p*} on the pointed loop space as follows:

- for $k \geq 2$,

$$\begin{aligned} \mu^{p*}(p(l) = k, \xi_1 = x_1, \tau_1 \in dt^1, \dots, \xi_k = x_k, \tau_k \in dt^k) \\ = \frac{1}{k} Q_{x_2}^{x_1} \dots Q_{x_1}^{x_k} (-L_{x_1}^{x_1}) e^{L_{x_1}^{x_1} t^1} dt^1 \dots (-L_{x_k}^{x_k}) e^{L_{x_k}^{x_k} t^k} dt^k. \end{aligned}$$

- for $k = 1$, $\mu^{p*}(p(\xi) = 1, \xi = x, \tau \in dt) = \frac{1}{t} e^{L_x^x t} dt$.

We call μ^{p*} the pointed loop measure.

Proposition 3.5. μ^{p*} induces the same loop measure as μ^b and μ^p .

Proof. It is obvious for the trivial loops. Let us focus on the non-trivial loops. For a non-trivial pointed loop $l = (\xi_1, \tau_1, \dots, \xi_n, \tau_n)$, define $\theta(l) = (\xi_2, \tau_2, \dots, \xi_n, \tau_n, \xi_1, \tau_1)$. Fix $n \geq 2$, $x_1, \dots, x_n \in S$, $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ measurable, define

$$\Phi(l) = 1_{\{p(l)=n\}} 1_{\{\xi_1=x_1, \dots, \xi_n=x_n\}} f(\tau_1, \dots, \tau_n)$$

and $\bar{\Phi} = \frac{1}{n}(\Phi + \Phi \circ \theta + \dots + \Phi \circ \theta^{n-1})$. By Proposition 3.3,

$$\mu^p(\bar{\Phi}) = \frac{1}{n} Q_{x_2}^{x_1} \dots Q_{x_1}^{x_n} \int_{\mathbb{R}_+^n} f(t^1, \dots, t^n) \left(\prod_{i=1}^n (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} t^i} dt^i \right).$$

From the definition of the pointed loop measure μ^{p*} , $\theta \circ \mu^{p*} = \mu^{p*}$,

$$\mu^{p*}(\bar{\Phi}) = \mu^{p*}(\Phi) = \frac{1}{n} Q_{x_2}^{x_1} \dots Q_{x_1}^{x_n} \int_{\mathbb{R}_+^n} f(t^1, \dots, t^n) \left(\prod_{i=1}^n (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} t^i} dt^i \right).$$

We have $\mu^p(\bar{\Phi}) = \mu^{p*}(\bar{\Phi})$. For a positive functional Φ on the space of pointed loops, we have the following decomposition

$$\Phi = \sum_{n \geq 1} \sum_{x \in S^n} 1_{\{p(l)=n\}} 1_{\{\xi_1=x_1, \dots, \xi_n=x_n\}} f^x(\tau_1, \dots, \tau_n)$$

where $f^x(\tau_1, \dots, \tau_n) = (\Phi|_{\{l: p(l)=n\}})(x_1, \tau_1, \dots, x_n, \tau_n)$. Define

$$\bar{\Phi} = \sum_{n \geq 1} \sum_{x \in S^n} \overline{1_{\{p(l)=n\}} 1_{\{\xi_1=x_1, \dots, \xi_n=x_n\}} f^x(\tau_1, \dots, \tau_n)}.$$

It is clear that $\bar{\cdot} : \Phi \rightarrow \bar{\Phi}$ is a well-defined linear map which preserves the positivity. By monotone convergence, $\mu^{p*}(\bar{\Phi}) = \mu^p(\bar{\Phi})$ for any positive measurable pointed loop functional. As a consequence, the loop measure induced by μ^{p*} is exactly μ . \square

Definition 3.12. For a pointed loop $l = (\xi_1, \tau_1, \dots, \xi_{p(l)}, \tau_{p(l)})$, $\xi = (\xi_1, \dots, \xi_{p(l)})$ is the corresponding discrete pointed loop. For any $F \subset S$, define $q(F, l) = \sum_{x \in F} N^x(l)$ the number of times l visits F . Recursively define the i -th hitting time for F as follows ($i = 1, \dots, q(F, l)$): $T_1^F(l) = T_1^F(\xi) = \inf\{m \leq p(l) : \xi_m \in F\}$ and $T_{i+1}^F(l) = T_{i+1}^F(\xi) = \inf\{m > T_i^F : m \leq p(l), \xi_m \in F\}$. Define $T = T_{q(F, l)}^F$ the last visiting time for F . Define $p(F, l) = \#\{i : \xi_{T_i^F} \neq \xi_{T_{i+1}^F}, i = 1, \dots, q(F, l)\}$ with the convention that $\xi_{T_{q(F, l)+1}^F} = \xi_{T_1^F}$. Define a pointed loop measure $\mu^{p*, F}$ as follows:

$$\begin{aligned} \mu^{p*, F} 1_{\{p(F, l) \neq 0\}} &= 1_{\{\xi_1 \in F, \xi_T \neq \xi_1\}} \frac{p(l)}{p(F, l)} \mu^{p*} \\ \mu^{p*, F} 1_{\{p(F, l) = 0\}} &= 1_{\{\xi_1 \in F, p(F, l) = 0\}} \frac{p(l)}{q(F, l)} \mu^{p*}. \end{aligned}$$

Remark 3. $p(F, l) = 0$ iff the intersection of the pointed loop l and the subset $F \subset S$ is a single element set: $|l \cap F| = 1$ (or $|\bigcup_{i=1}^{q(F,l)} \{\xi_{T_i^F}\}| = 1$ equivalently). For a loop l with $p(F, l) \neq 0$ (or $p(F, l) = 2, \dots, \infty$ equivalently), the term $1_{\{\xi_1 \in F, \xi_T \neq \xi_1\}}$ in the above expression implies that $\mu^{p^*, F}|_{\{l: p(F, l) \neq 0\}}$ is concentrated on the pointed loops satisfying the following two conditions:

1. the pointed loop starts from a point in F .
2. the trace of the pointed loop on F has an endpoint different from the starting point.

By an argument similar to remark 3.5, it can be showed that $\mu^{p^*, F}$ induces a loop measure which is exactly the restriction of μ to the loops visiting F .

Definition 3.13 (Multi-occupation field). Define the circular permutation operator r_j as follows: $r_j(z^1, \dots, z^p) = (z^{1+j}, \dots, z^n, z^1, \dots, z^j)$. For any $f : S^n \rightarrow \mathbb{R}$ measurable, define the multi-occupation field of a based loop l of length t as

$$\langle l, f \rangle = \sum_{j=0}^{n-1} \int_{0 < s^1 < \dots < s^n < t} f \circ r_j(l(s_1), \dots, l(s_n)) ds^1 \dots ds^n.$$

If l_1 and l_2 are two equivalent based loops, they correspond to the same multi-occupation field. Therefore, it is well-defined for loops. When $n = 1$, it is called the occupation time. For $x \in \mathbb{R}^m$ for some integer m , define $l^x = \langle l, \delta_x \rangle$ where $\delta_x(y) = 1_{\{x=y\}}$.

Definition 3.14 (Another bridge measure $\mu^{x,y}$). Another bridge measure $\mu^{x,y}$ can be defined on paths from x to y :

$$\mu^{x,y}(d\gamma) = \int_0^\infty \mathbb{P}_t^{x,y}(d\gamma) dt.$$

For a path γ from x to y , let $p(\gamma)$ be the total number of jumps, T_i the i -th jumping time and T the time duration of γ . Then γ can be viewed as $(x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = \gamma(T_{p(\gamma)}), T - T_{p(\gamma)})$.

The bridge measure $\mu^{x,y}$ can be expressed as follows:

Proposition 3.6.

$$\begin{aligned} \mu^{x,y}(p(\gamma) = p, \gamma(T_1) = x_1, \dots, \gamma(T_{p-1}) = x_{p-1}, \\ T_1 \in dt^1, T_2 - T_1 \in dt^2, \dots, T_p - T_{p-1} \in dt^p, T - T^p \in dt^{p+1}) \\ = Q_{x_1}^x Q_{x_2}^{x_1} \dots Q_{y}^{x_{p-1}} 1_{\{t^1, \dots, t^{p+1} > 0\}} (-L_x^x) e^{L_x^{x_1} t^1} (-L_{x_1}^{x_1}) e^{L_{x_1}^{x_1} t^2} \dots (-L_{x_{p-1}}^{x_{p-1}}) e^{L_{x_{p-1}}^{x_{p-1}} t^p} e^{L_y^y t^{p+1}} \prod_{j=1}^{p+1} dt^j \end{aligned}$$

In the case $x = y$, $\gamma = (x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = T_{p(\gamma)}, T - T_{p(\gamma)})$ can be viewed as a based loop. Therefore, $\mu^{x,x}$ can be viewed as a measure on the based loop. Moreover, $\mu^{x,x}(dl) = 1_{\{l(0)=x\}}|l|\mu^b(dl)$. Consequently, the loop measure induced by $\mu^{x,x}$, which will be denoted by the same notation $\mu^{x,x}$, has the following relation with the loop measure μ .

Proposition 3.7.

$$\mu^{x,x}(dl) = l^x \mu(dl).$$

In the case $x \neq y$, $\gamma = (x, T_1, \gamma(T_1), T_2 - T_1, \gamma(T_2), \dots, T_{p(\gamma)} - T_{p(\gamma)-1}, y = \gamma(T_{p(\gamma)}), T - T_{p(\gamma)})$ can be viewed as a pointed loop. Similarly, $\mu^{x,y}$ can be viewed as a measure on the pointed loop. Moreover, $L_x^y \mu^{x,y}(dl) = 1_{\{l \text{ starts at } x \text{ and ends up at } y\}} p(l) \mu^{p*}(dl)$. Consequently, the loop measure induced by $\mu^{x,y}$, which will be denoted by the same notation $\mu^{x,y}$, has the following relation with the loop measure μ .

Proposition 3.8.

$$L_x^y \mu^{x,y}(dl) = N_x^y \mu(dl).$$

3.2 Compatibility of the loop measure with time change

Proposition 3.9. *Suppose $\lambda : S \rightarrow [0, \infty]$. Given a Markov process $(X_t, t \geq 0)$ in S , define $B_t = \int_0^t \lambda(X_s) ds$. Let $(C_t, t \geq 0)$ be the right-continuous inverse of $(B_t, t \geq 0)$. Define $\zeta = \inf\{s \geq 0 : C_s = C_\infty\}$. Define $Y_t = X_{C_t}, t < \zeta$ (it will be called the time-changed process of X with respect to λ and denoted $\lambda(X)$). On the space of based (pointed) loops contained in $\{x \in S : \lambda(x) < \infty\}$, λ defines a similar operation. If l_1 and l_2 are two equivalent based (pointed) loops, $\lambda(l_1)$ and $\lambda(l_2)$ are equivalent again. Consequently, λ can be defined on the space of loops with the domain $D(\lambda) = \{\text{loops contained in } \{x \in S : \lambda(x) < \infty\}\}$. There are two Markovian loop measures μ_X, μ_Y defined by X, Y respectively. The following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & Y \\ \downarrow & & \downarrow \\ \mu_X & \xrightarrow{\lambda} & \mu_Y \end{array}$$

In particular, the loop measure is compatible with the notion of "trace on a set" (i.e. $\lambda = 1_A$) and "restriction" (i.e. $\lambda = 1_A + \infty \cdot 1_{A^c}$).

Proof. Let $\lambda \circ \mu$ be the image law of μ under the mapping λ . Denote by $\pi^{p \rightarrow o}$ the quotient map from pointed loops to loops. Then, we have to show that λ commutes with $\pi^{p \rightarrow o}$.

The holding times are almost surely different for μ_X, μ_Y and $\lambda \circ \mu_X$. So the same is true

for the measures on pointed loops μ_X^{p*}, μ_Y^{p*} and $\lambda \circ \mu_X^{p*}$.

Every change of time can be done in three steps: i) Restriction, ii) trace, iii) time change with a function $0 < \lambda < \infty$. Accordingly, it is enough to deal with these three special cases separately:

i) $0 < \lambda < \infty$

Let L and \hat{L} represent the generator of X and Y . Then $\hat{L}_y^x = \frac{L_y^x}{\lambda_x}$.

By Definition 3.11 and its following remark,

$$\begin{aligned} \lambda \circ \mu_X^{p*}(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \tau_1 \in dt^1, \dots, \tau_k \in dt_k) \\ = \mu_X^{p*}(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \lambda_{x_1}\tau_1 \in dt^1, \dots, \lambda_{x_k}\tau_k \in dt_k) \\ = \frac{1}{k} L_{x_2}^{x_1} \dots L_{x_1}^{x_k} e^{L_{x_1}^{x_1} t^1 / \lambda_{x_1}} \dots e^{L_{x_k}^{x_k} t^k / \lambda_{x_k}} dt^1 \dots dt^k \\ = \frac{1}{k} \hat{L}_{x_2}^{x_1} \dots \hat{L}_{x_1}^{x_k} e^{\hat{L}_{x_1}^{x_1} t^1} \dots e^{\hat{L}_{x_k}^{x_k} t^k} dt^1 \dots dt^k \\ = \mu_Y^{p*}(p(\xi) = k, \xi_1 = x_1, \dots, \xi_k = x_k, \tau_1 \in dt^1, \dots, \tau_k \in dt_k) \end{aligned}$$

Therefore, $\lambda \circ \mu_X = \lambda \circ \pi^{p \rightarrow o} \circ \mu_X^{p*} = \pi^{p \rightarrow o} \circ \lambda \circ \mu_X^{p*} = \pi^{p \rightarrow o} \mu_Y^{p*} = \mu_Y$

ii) $\lambda = 1_A + \infty \cdot 1_{A^c}$. In that case, $\lambda \circ \mu_X = \mu_X|_{D(\lambda)} = \mu_Y$.

iii) $\lambda = 1_A + 0 \cdot 1_{A^c}$.

We need to show that $\lambda \circ \mu_X = \mu_Y$. We will only prove this for the non-trivial loops. The trivial loop case can be proved in a similar way.

Use \mathbb{P}^x to stand for the law of a minimal Markov process X starting from x . Let T_1 be the first jumping time, and set $T_{1,A} = \inf\{s \geq T_1, X_s \in A\}$. Let $(R^A)_y^x = \mathbb{P}^x[X_{T_{1,A}} = y]$ for $x, y \in S$. Obviously, $(R^A)_y^x = 0$ for $y \in A^c$. By Proposition 2.6, the relation between the generator L of X and the generator \hat{L} of Y is stated as follows: $\hat{L}_x^x = L_x^x(1 - (R^A)_x^x)$, $\hat{L}_y^x = -(R^A)_y^x L_x^x$ for $x \neq y$.

Fix a non-trivial discrete pointed loop (x_1, \dots, x_n) where $x_i \in A$ for $i = 1, \dots, n$. Take $F = \bigcup_{i=1}^n \{x_i\}$. Take n positive measurable functions f_1, \dots, f_n on S . By Definition 3.12 and its following remark, it is enough to show that

$$\begin{aligned} \lambda \circ \mu_X^{p*,F}(p(\xi) = n, \xi_1 = x_1, \dots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i)) \\ = \mu_Y^{p*}(p(\xi) = n, \xi_1 = x_1, \dots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i)). \end{aligned}$$

In order that $\lambda(l)$, the image of the pointed loop l , equals $(p(\xi) = n, x_1, \tau_1, \dots, x_n, \tau_n)$, the pointed loop l has to be of the following form $\mu_X^{p*,F}$ -a.s.:

$$(\xi_{111}, \tau_{111}, \dots, \xi_{11M_1^1}, \tau_{11M_1^1}, \dots, \xi_{1N_11}, \tau_{1N_11}, \dots, \xi_{1N_1M_{N_1}^1}, \tau_{1N_1M_{N_1}^1},$$

$$\begin{aligned}
& \xi_{211}, \tau_{211}, \dots, \xi_{21M_1^2}, \tau_{21M_1^2}, \dots, \xi_{2N_21}, \tau_{2N_21}, \dots, \xi_{2N_2M_{N_2}^2}, \tau_{2N_2M_{N_2}^2}, \\
& \dots \\
& \xi_{n11}, \tau_{n11}, \dots, \xi_{n1M_1^n}, \tau_{n1M_1^n}, \dots, \xi_{nN_n1}, \tau_{nN_n2}, \dots, \xi_{nN_nM_{N_n}^n}, \tau_{nN_nM_{N_n}^n}
\end{aligned}$$

with

- $\xi_{ij1} = x_i$ for all i, j ;
- $\xi_{ijk} \in A^c$ for $k \neq 1$ and all i, j ;
- $\tau_i = \sum_j \tau_{ij1}$.

Roughly speaking, $\xi_{ij1}, \tau_{ij1}, \dots, \xi_{ijM_j^i}, \tau_{ijM_j^i}$ can be viewed as an excursion in A^c from x_i to x_i for $j \neq N_i$. And $\xi_{iN_i1}, \tau_{iN_i1}, \dots, \xi_{iN_iM_{N_i}^i}, \tau_{iN_iM_{N_i}^i}$ can be viewed as an excursion in A^c from x_i to x_{i+1} . Accordingly,

$$\begin{aligned}
\lambda \circ \mu_X^{p*,F}(p(\xi) = n, \xi_1 = x_1, \dots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i)) \\
= \sum_{\xi} \mu_X^{p*,F}(\xi_{ij1} = x_i \text{ and for all } i, j, \xi_{ijn} \in A^c \text{ for } n \neq 1, \prod_{i=1}^n f_i(\sum_j \tau_{ij1})).
\end{aligned}$$

Since $Q_y^x + \sum_{p=1}^{\infty} \sum_{z_1, \dots, z_p \in A^c} Q_{z_1}^x Q_{z_2}^{z_1} \dots Q_{z_p}^{z_{p-1}} Q_y^{z_p} = (R^A)_y^x$, the above quantity equals

$$\begin{aligned}
& \frac{1}{n} \sum_{N_1, \dots, N_n \geq 1} \prod_{i=1}^n ((R^A)_{x_i}^{x_i})^{N_i-1} (R^A)_{x_{i+1}}^{x_i} \\
& \int f(t^{i11} + \dots + t^{iN_i1}) (-L_{x_i}^{x_i})^{N_i} e^{L_{x_i}^{x_i}(t^{i11} + \dots + t^{iN_i1})} dt^{i11} \dots dt^{iN_i1} \\
& = \frac{1}{n} \sum_{N_1, \dots, N_k \geq 1} \prod_{i=1}^n \int ((R^A)_{x_i}^{x_i})^{N_i-1} (R^A)_{x_{i+1}}^{x_i} \frac{(t^i)^{N_i-1}}{(N_i-1)!} (-L_{x_i}^{x_i})^{N_i} e^{L_{x_i}^{x_i} t^i} f(t^i) dt^i \\
& = \frac{1}{n} \prod_{i=1}^n \int -L_{x_i}^{x_i} (R^A)_{x_{i+1}}^{x_i} e^{L_{x_i}^{x_i} t^i (1 - (R^A)_{x_i}^{x_i})} f(t^i) dt^i \\
& = \frac{1}{n} \prod_{i=1}^n \int (L_A)_{x_{i+1}}^{x_i} e^{(L_A)_{x_i}^{x_i} t^i} f(t^i) dt^i \\
& = \mu_Y^{p*}(p(\xi) = n, \xi_1 = x_1, \dots, \xi_n = x_n, \prod_{i=1}^n f_i(\tau_i)) \text{ for } n \geq 2.
\end{aligned}$$

For $n = 1$, it can be proved in a similar way. Finally, we conclude that $\lambda \circ \mu_X = \mu_Y$. □

3.3 Decomposition of the loops and excursion theory

Fix some set $F \subset S$.

Definition 3.15 (excursion outside F). By non-empty excursion outside F , we mean a multiplet of the form $((\xi_1, \tau^1, \dots, \xi_k, \tau^k), A, B)$ for some $k \in \mathbb{N}_+$, $\xi_1, \dots, \xi_k \in F^c$, $A, B \in F$ and $\tau^1, \dots, \tau^k \in \mathbb{R}_+$. Let $T_0 = 0$ and $T_m = \tau^1 + \dots + \tau^m$ for $m = 1, \dots, k$. Define $e : [0, T_k[\rightarrow F^c$ such that $e(u) = \xi_m$ for $u \in [T_{m-1}, T_m[$. Therefore, the excursion can be viewed as a path e attached to starting point A and ending point B and it will also be denoted by (e, A, B) . By empty excursion, we mean (ϕ, A, B) .

Definition 3.16 (excursion measure outside F). Define a family of probability measure $\nu_{F,ex}^{x,y}$ indexed by $x, y \in F$ as follows:

$$\begin{aligned} \nu_{F,ex}^{x,y}(\xi_1 = x_1, \tau^1 \in dt^1, \dots, \xi_k = x_k, \tau^k \in dt^k, A = u, B = v) \\ = \delta_{(u,v)}^{(x,y)} \frac{1}{(R^F)_y^x} 1_{\{x_1, \dots, x_k \in F^c\}} Q_{x_1}^x L_{x_2}^{x_1} \dots L_{x_k}^{x_{k-1}} L_y^{x_k} e^{L_{x_1}^{x_1} t^1} \dots e^{L_{x_k}^{x_k} t^k} dt^1 \dots dt^k. \end{aligned}$$

and $\nu_{F,ex}^{x,y}[(\phi, A, B) = (\phi, u, v)] = \delta_{(u,v)}^{(x,y)} \frac{Q_y^x}{(R^F)_y^x}$. Recall that

$$(R^F)_y^x = \begin{cases} Q_y^x + \sum_{k \geq 1} \sum_{x_1, \dots, x_k \in F^c} Q_{x_1}^x Q_{x_2}^{x_1} \dots Q_{x_k}^{x_{k-1}} Q_y^{x_k} & \text{for } y \in F \\ 0 & \text{otherwise.} \end{cases}$$

Define a function $\phi^{br \rightarrow ex}$ from the space of bridges to the space of excursions as follows: Given a bridge γ from x to y , which is represented by

$$(x, T_1, \gamma(T_1), T_2 - T_1, \dots, \gamma(T_{p(\gamma)-1}), T_{p(\gamma)} - T_{p(\gamma)-1}, y = \gamma(T_{p(\gamma)}), T - T_{p(\gamma)}),$$

we represent $\phi^{br \rightarrow ex}(\gamma)$ by

$$((\gamma(T_1), T_2 - T_1, \dots, \gamma(T_{p(\gamma)-1}), T_{p(\gamma)} - T_{p(\gamma)-1}), x, y).$$

The image measure of $\mu^{x,y}$ under $\phi^{br \rightarrow ex}$, namely $\phi^{br \rightarrow ex} \circ \mu^{x,y}$, has the following relation with the excursion measure $\nu_{F,ex}^{x,y}$:

Proposition 3.10.

$$\nu_{F,ex}^{x,y}(d\gamma) = \frac{1}{-L_y^y R_y^x} \phi^{br \rightarrow ex} \circ \mu^{x,y}(d\gamma, \gamma(T_1), \dots, \gamma(T_{p(\gamma)-1}) \in F^c)$$

Define a function $\varphi_F^{ex \rightarrow po}$ from the space of non-empty excursions out of F to the space of pointed loops as follows:

$$\varphi_F^{ex \rightarrow po} : ((\xi_1, \tau^1, \dots, \xi_k, \tau^k), A, B) \rightarrow (\xi_1, \tau^1, \dots, \xi_k, \tau^k)$$

Accordingly, $\nu_{F,ex}^{x,y}$ induces a pointed loop measure on the space of pointed loops outside of F , which is denoted by the same notation $\nu_{F,ex}^{x,y}$. The relation with the pointed loop measure is as follows:

Proposition 3.11. Let $C = \{(\xi_1, \tau^1, \dots, \xi_n, \tau^n) \in \{\text{pointed loops}\} : Q_{\xi_1}^{\xi_n} > 0\}$. Then, $\varphi_F^{ex \rightarrow po} \circ \nu_{F,ex}^{x,y}$ is absolutely continuous with respect to μ^{p*} . Moreover,

$$Q_{\xi_1}^{\xi_n} \frac{d\varphi_F^{ex \rightarrow po} \circ \nu_{F,ex}^{x,y}}{d\mu^{p*}}((\xi_1, \tau^1, \dots, \xi_n, \tau^n)) = 1_{\{R_y^x > 0, \xi_1, \dots, \xi_n \in F^c\}} \frac{n Q_{\xi_1}^x Q_y^{\xi_n}}{R_y^x}$$

Definition 3.17 (Decomposition of a loop). Let $l = (\xi_1, \tau^1, \dots, \xi_k, \tau^k)^o$ be a loop visiting F . The pre-trace of the loop l on F is obtained by removing all the ξ_m, τ^m such that $\xi_m \in F^c$ for $m = 1, \dots, k$. We denote it by $Ptr_F(l)$. Suppose the pre-trace on F can be written as $(\mathfrak{x}_1, \mathfrak{s}^1, \dots, \mathfrak{x}_q, \mathfrak{s}^q)^o$. Then we can write the loop l^o in the following form:

$$(\mathfrak{x}_1, \mathfrak{s}^1, y_1^1, t_1^1, \dots, y_{m_1}^1, t_{m_1}^1, \mathfrak{x}_2, \mathfrak{s}^2, y_1^2, t_1^2, \dots, y_{m_2}^2, t_{m_2}^2, \dots, \mathfrak{x}_q, \mathfrak{s}^q, y_1^q, t_1^q, \dots, y_{m_q}^q, t_{m_q}^q)^o$$

with $\mathfrak{x}_i \in F$ for all i and $y_j^i \in F^c$ for all i, j (with the following convention: if $m_i = 0$ for some $i = 1, \dots, q$, $y_1^i, t_1^i, \dots, y_{m_i}^i, t_{m_i}^i$ does not appear in the above expression). We will use e_i to stand for $(y_1^i, t_1^i, \dots, y_{m_i}^i, t_{m_i}^i)$ with the convention that $e_i = \phi$ if $m_i = 0$. Define a point measure $\mathcal{E}_F(l) = \sum_i \delta_{(e_i, \mathfrak{x}_i, \mathfrak{x}_{i+1})}$. Define $N_y^x(Ptr_F(l)) = \sum_{i=1}^q 1_{\{\mathfrak{x}_i = x, \mathfrak{x}_{i+1} = y\}}$ with the convention that $\mathfrak{x}_{q+1} = \mathfrak{x}_1$. Set $q(Ptr_F(l)) = \sum_{x,y} N_y^x(Ptr_F(l))$. In particular, in the case above, we have $q(Ptr_F(l)) = q$ if $q \geq 2$ and $q(Ptr_F(l)) = 0$ if $q = 1$.

Remark 4. The pre-trace $(\mathfrak{x}_1, \mathfrak{s}^1, \dots, \mathfrak{x}_q, \mathfrak{s}^q)^o$ of a loop l on F is not necessarily a loop. We allow $\mathfrak{x}_i = \mathfrak{x}_{i+1}$ for some $i = 1, \dots, q$ which is prohibited in the definition we gave of a loop.

Definition 3.18. The pre-trace of a loop l on F can always be written as follows:

$$(\mathfrak{x}_1, \mathfrak{s}_1^1, \dots, \mathfrak{x}_1, \mathfrak{s}_{m_1}^1, \mathfrak{x}_2, \mathfrak{s}_1^2, \dots, \mathfrak{x}_2, \mathfrak{s}_{m_2}^2, \dots, \mathfrak{x}_k, \mathfrak{s}_1^k, \dots, \mathfrak{x}_k, \mathfrak{s}_{m_k}^k)^o$$

with $\mathfrak{x}_i \neq \mathfrak{x}_{i+1}$ for $i = 1, \dots, k$ with the usual convention that $\mathfrak{x}_{k+1} = \mathfrak{x}_1$. Then, l_F , the trace of l on F is defined by

$$(\mathfrak{x}_1, \mathfrak{t}^1 = \mathfrak{s}_1^1 + \dots + \mathfrak{s}_{m_1}^1, \mathfrak{x}_2, \mathfrak{s}_1^2 + \dots + \mathfrak{s}_{m_2}^2, \dots, \mathfrak{x}_k, \mathfrak{t}^k = \mathfrak{s}_1^k + \dots + \mathfrak{s}_{m_k}^k)^o.$$

Formally, the trace of l on F is obtained by throwing away the parts out of F and then by gluing the rest in circular order.

By replacing μ by $\mu^{p*,F}$ and considering the pointed loops, we have the following propositions.

Proposition 3.12. Let f be some measurable positive function on the space of excursions and g a positive measurable function on the space of pre-traces on F . Then,

$$\mu(1_{\{l \text{ visits } F\}} g(Ptr_F(l)) e^{-(\mathcal{E}_F(l), f)}) = \mu(1_{\{l \text{ visits } F\}} g(Ptr_F(l)) \prod_{x,y \in F} (\nu_{F,ex}^{x,y}(e^{-f}))^{N_y^x(Ptr_F(l))}).$$

Proposition 3.13. *The image measure $\mu_{Ptr,F}^{p*}$ of the pointed loop measure $\mu^{p*,F}$ under the map of the pre-trace on F can be described as follows:*

- if x_1, \dots, x_q are not identical, then

$$\begin{aligned} \mu_{Ptr,F}^{p*}(q(Ptr_F(l)) = q, \mathfrak{x}_1 = x_1, \mathfrak{s}^1 \in ds^1, \dots, \mathfrak{x}_q = x_q, \mathfrak{s}^q \in ds^q) \\ = \frac{1}{p(l_F)} \prod_{x,y} ((R^F)_y^x)^{N_y^x(Ptr_F(l))} \prod_{i=1}^q (-L_{x_i}^{x_i}) e^{L_{x_i}^{x_i} s^i} ds^i; \end{aligned}$$

- if $x_1 = \dots = x_q = x$ and $q > 1$, then

$$\begin{aligned} \mu_{Ptr,F}^{p*}(q(Ptr_F(l)) = q, \mathfrak{x}_1 = x_1 = \dots = \mathfrak{x}_q = x_q = x, \mathfrak{s}^1 \in ds^1, \dots, \mathfrak{s}^q \in ds^q) \\ = \frac{1}{q} ((R^F)_x^x)^q \prod_{i=1}^q (-L_x^x) e^{L_x^x s^i} ds^i \end{aligned}$$

- if $q = 1$ and $x_1 = x$, then

$$\begin{aligned} \mu_{Ptr,F}^{p*}(q(Ptr_F(l)) = 1, \mathfrak{x} = x, \mathfrak{s} \in ds) = & \underbrace{(R^F)_x^x (-L_x^x) e^{L_x^x s} ds}_{\text{contribution of the non-trivial loops}} \\ & + \underbrace{\frac{1}{s} e^{L_x^x s} ds}_{\text{contribution of the trivial loops}} \end{aligned}$$

where $\mu_{Ptr,F}^{p*}$ is the image measure of the pointed loop measure $\mu^{p*,F}|_{\{\text{loops visiting } F\}}$.

Proposition 3.14. *Under the same notation as Definition 3.18,*

- for $k > 1$,

$$\begin{aligned} \mu_{Ptr,F}^{p*}(\mathfrak{x}_1 = x_1, \dots, \mathfrak{x}_k = x_k, m_1 = q_1, \dots, m_k = q_k, \mathfrak{t}^1 \in dt^1, \dots, \mathfrak{t}^k \in dt^k) \\ = \frac{1}{k} (L_F)_{x_2}^{x_1} \dots (L_F)_{x_1}^{x_k} \prod_{i=1}^k e^{(L_F)_{x_i}^{x_i} t^i} dt^i \prod_{j=1}^k e^{(L_{x_j}^{x_j} - (L_F)_{x_j}^{x_j}) t^j} \frac{((-L_{x_j}^{x_j} + (L_F)_{x_j}^{x_j}) t^j)^{q_j-1}}{(q_j - 1)!} \end{aligned}$$

- for $k = 1$, $q_1 = q > 1$,

$$\begin{aligned} \mu_{Ptr,F}^{p*}(\mathfrak{x}_1 = x_1, m_1 = q_1, \mathfrak{t}^1 \in dt^1) \\ = \frac{1}{t_1} e^{(L_F)_{x_1}^{x_1} t^1} dt^1 e^{(L_{x_1}^{x_1} - (L_F)_{x_1}^{x_1}) t^1} \frac{((-L_{x_1}^{x_1} + (L_F)_{x_1}^{x_1}) t^1)^{q_1}}{q_1!} \end{aligned}$$

- for $k = 1$ and $q_1 = 1$,

$$\begin{aligned} \mu_{Ptr,F}^{p*}(\mathfrak{x}_1 = x_1, m_1 = 1, \mathfrak{t}^1 \in dt^1) \\ = \frac{1}{t_1} e^{(L_F)_{x_1}^{x_1} t^1} dt^1 e^{(L_{x_1}^{x_1} - (L_F)_{x_1}^{x_1}) t^1} ((-L_{x_1}^{x_1} + (L_F)_{x_1}^{x_1}) t^1) + \frac{1}{t_1} e^{L_{x_1}^{x_1} t^1} dt^1 \end{aligned}$$

Proof. The result comes from Proposition 3.13 and Proposition 2.6. \square

Combining Proposition 3.12 and Proposition 3.14, we have the following proposition:

Proposition 3.15.

$$\begin{aligned} & \mu(1_{\{l \text{ visits } F\}} g(l_F) e^{-\langle \mathcal{E}_F(l), f \rangle}) \\ &= \mu(1_{\{l \text{ visits } F\}} g(l_F) \prod_{x \neq y \in F} \nu_{F,ex}^{x,y}(e^{-f})^{N_y^x(l_F)} e^{\sum_{x \in F} (L_x^x - (L_F)_x^x) l_F^x \nu_{F,ex}^{x,x} (1 - e^{-f})}) \end{aligned}$$

Corollary 3.16. *We see that $\nu_{F,ex}^{x,y}$ is a probability measure on the space of the excursions from x to y out of F . By mapping an excursion (e, x, y) into the Dirac measure $\delta_{(e,x,y)}$, $\nu_{F,ex}^{x,y}$ induces a probability measure on $\mathcal{M}^p(\{\text{excursions}\})$, the space of point measures over the space of excursions. We will adopt the same notation $\nu_{F,ex}^{x,y}$. Choose k samples of the excursions according to $\nu_{F,ex}^{x,y}$, namely ex_1, \dots, ex_k , then $\sum_i \delta_{ex_i}$ has the law $(\nu_{F,ex}^{x,y})^{\otimes k}$. For any $\beta = (\beta^x, x \in F) \in \mathbb{R}_+^F$, let $\mathcal{N}_F(\beta)$ be a Poisson random measure on the space of excursions with intensity $\sum_x (-L_x^x + (L_F)_x^x) \beta^x \nu_{F,ex}^{x,x}$. Let $l_F \rightarrow K(l_F, \cdot)$ be a transition kernel from $\{\text{the trace of the loop on } F\}$ to $\{\text{point measure over the space of excursions}\}$ as follows:*

$$K(l_F, \cdot) = \bigotimes_{x \neq y \in F} (\nu_{F,ex}^{x,y})^{\otimes N_y^x(l_F)} \bigotimes \mathcal{N}_F((l_F^x, x \in F))$$

Then the joint measure of $(l_F, \mathcal{E}_F(l))$ is $\mu_F(dl_F) K(l_F, \cdot)$ where μ_F is the image measure of μ under $l \rightarrow l_F$. By Proposition 3.9, μ_F is actually the loop measure associated with the trace of the Markov process on F or with L_F equivalently.

Remark 5. $K(l_F, \cdot)$ can also be viewed as a Poisson random measure on the space of excursions with intensity $\sum_x (-L_x^x + (L_F)_x^x) l_F^x \nu_{F,ex}^{x,x} + \sum_{x \neq y \in F} \nu_{F,ex}^{x,y}$ conditioned to have exactly $N_y^x(l_F)$ excursions from x to y out of F for all $x \neq y \in F$.

Definition 3.19. Suppose χ is a non-negative function on S vanishing on F . For an excursion (e, A, B) , define the real-valued function $\langle \chi, \cdot \rangle$ of the excursion as follows:

$$\langle \chi, (e, A, B) \rangle = \int \chi(e(t)) dt.$$

Lemma 3.17. *We see that the excursion measure $\nu_{F,ex}^{x,y}$ varies as the generator changes. Let $\nu_{F,ex}^{x,y,\chi}$ be the excursion measure when L is replaced by $L - M_\chi$. Define $(R_\chi^F)_y^x$ as $(R^F)_y^x$ when L is replaced by $L - M_\chi$. Then,*

$$e^{-\langle \chi, \cdot \rangle} \cdot \nu_{F,ex}^{x,y} = \frac{(R_\chi^F)_y^x}{(R^F)_y^x} \nu_{F,ex}^{x,y,\chi}.$$

In particular,

$$\nu_{F,ex}^{x,y}[e^{-\langle \chi, \cdot \rangle}] = \frac{(R_\chi^F)_y^x}{(R^F)_y^x}.$$

Accordingly, we have the following corollary,

Corollary 3.18.

$$\begin{aligned} & \mu(1_{\{l \text{ visits } F\}} g(l_F) e^{-\sum_{\mathcal{E}_F(l)} \langle \chi, \cdot \rangle}) \\ &= \mu \left(1_{\{l \text{ visits } F\}} g(l_F) \prod_{x \neq y \in F} \left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x} \right)^{N_y^x(l_F)} e^{\sum_{x \in F} (L_x^x - (L_F)_x^x) l_F^x \left(1 - \frac{(R_\chi^F)_x^x}{(R^F)_x^x} \right)} \right). \end{aligned}$$

3.4 Further properties of the multi-occupation field

We know the loop measure varies as the generator varies. To emphasize this, we write $\mu(L, dl)$ instead of $\mu(dl)$.

Proposition 3.19. $e^{-\langle l, \chi \rangle} \mu(L, dl) = \mu(L - M_\chi, dl)$ for positive measurable function χ on S .

Proof. It is the direct consequence of the Feynman-Kac formula. To be more precise,

$$\begin{aligned} e^{-\langle l, \chi \rangle} \mu(L, dl) &= \sum_{x \in S} \int \frac{1}{t} \mathbb{P}_{t,x}^x [e^{-\langle l, \chi \rangle}, dl] dt = \sum_{x \in S} \int \frac{1}{t} \mathbb{P}_t^x [e^{-\langle l, \chi \rangle} 1_{\{l(t)=x\}}, dl] dt \\ &= \sum_{x \in S} \int \frac{1}{t} \mathbb{P}_{(\chi)_t}^x [1_{\{l(t)=x\}}, dl] dt = \mu(L - M_\chi, dl). \end{aligned}$$

□

Proposition 3.20. Suppose $f : S^n \rightarrow \mathbb{R}$ is positive measurable, then

$$\mu(\langle l, f \rangle) = \sum_{(y_1, \dots, y_n) \in S^n} V_{y_2}^{y_1} \cdots V_{y_1}^{y_n} f(y_1, \dots, y_n).$$

Proof.

$$\begin{aligned} \mu(\langle l, f \rangle) &= \int_0^\infty \frac{1}{t} \sum_{x \in S} \mathbb{P}_{t,x}^x \left[\sum_{j=0}^{n-1} \int_{0 < s^1 < \dots < s^n < t} f \circ r_j(l(s^1), \dots, l(s^n)) \prod_{i=1}^n ds^i \right] dt \\ &= \int_{0 < s^1 < \dots < s^n < t < \infty} \sum_{j=1}^n \sum_{(x, x_1, \dots, x_n) \in S^{n+1}} ds^1 \cdots ds^n dt \\ &\quad \frac{1}{t} f \circ r_j(x_1, \dots, x_n) (P_{s^1}^x)_{x_1} (P_{s^2-s^1}^{x_2})_{x_1}^{x_2} \cdots (P_{t-s^n})_x^{x_n} \\ &= \int_{0 < s^1 < \dots < s^n < t < \infty} \sum_{j=1}^n \sum_{(x_1, \dots, x_n) \in S^n} ds^1 \cdots ds^n dt \\ &\quad \frac{1}{t} f \circ r_j(x_1, \dots, x_n) (P_{s^2-s^1})_{x_2}^{x_1} (P_{s^3-s^2})_{x_3}^{x_2} \cdots (P_{t-s^n+s^1})_{x_1}^{x_n}. \end{aligned}$$

Performing the change of variables $a^0 = s^1, a^1 = s^2 - s^1, \dots, a^{n-1} = s^n - s^{n-1}, a^n = t - s^n + s^1$,

$$\begin{aligned}\mu(\langle l, f \rangle) &= \int_{a^0, \dots, a^n > 0, a^n > a^0} da^0 \cdots da^n \sum_{(x_1, \dots, x_n) \in S^n} \sum_{j=0}^{n-1} \frac{1}{a^1 + \cdots + a^n} \\ &\quad f \circ r_j(x_1, \dots, x_n) (P_{a^1})_{x_2}^{x_1} \cdots (P_{a^n})_{x_1}^{x_n} \\ &= \int_{a^1, \dots, a^n > 0} da^1 \cdots da^n \sum_{(x_1, \dots, x_n) \in S^n} \sum_{j=0}^{n-1} \frac{a_n}{a^1 + \cdots + a^n} \\ &\quad f \circ r_j(x_1, \dots, x_n) (P_{a^1})_{x_2}^{x_1} \cdots (P_{a^n})_{x_1}^{x_n}.\end{aligned}$$

Changing again variables with $b^{1+j} = a^1, \dots, b^n = a^{n-j}, b^1 = a^{n-j+1}, \dots, b^j = a^n$ and $y_{1+j} = x_1, \dots, y_n = x_{n-j}, y_1 = x_{n-j+1}, \dots, y_j = x_n$, and summing the integrals for all j ,

$$\begin{aligned}\langle l, f \rangle &= \int_{b^1, \dots, b^n > 0} \sum_{(y_1, \dots, y_n) \in S^n} (P_{b^1})_{y_2}^{y_1} \cdots (P_{b^n})_{y_1}^{y_n} f(y_1, \dots, y_n) db^1 \cdots db^n \\ &= \sum_{(y_1, \dots, y_n) \in S^n} V_{y_2}^{y_1} \cdots V_{y_1}^{y_n} f(y_1, \dots, y_n).\end{aligned}$$

□

Define $\tilde{\mathfrak{S}}_{n,m} \subset \mathfrak{S}_{n+m}$ to be the collection of permutations σ on $\{1, \dots, n+m\}$ such that the order of $1, \dots, n$ and $n+1, \dots, n+m$ is preserved under the permutation σ respectively, i.e.

$$\tilde{\mathfrak{S}}_{n,m} = \{\sigma \in \mathfrak{S}_{n+m} : \sigma(1) < \cdots < \sigma(n) \text{ and } \sigma(n+1) < \cdots < \sigma(n+m)\}.$$

Define $\mathfrak{S}_{n,m}^1 = \{\sigma \in \mathfrak{S}_{n,m} ; \sigma(1) = 1\}$. Then, we have $\sigma(1) < \cdots < \sigma(n)$ for $\sigma \in \mathfrak{S}_{n,m}^1$.

Proposition 3.21 (Shuffle product). *Suppose $f : S^n \rightarrow \mathbb{R}, g : S^m \rightarrow \mathbb{R}$ bounded or positive and measurable. Then,*

$$\langle l, f \rangle \langle l, g \rangle = \sum_{j=0}^{m-1} \sum_{\sigma \in \tilde{\mathfrak{S}}_{n,m}} \langle l, (f \otimes (g \circ r_j)) \circ \sigma^{-1} \rangle.$$

Proof. Let t be the length of l .

$$\begin{aligned}\langle l, f \rangle \langle l, g \rangle &= \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \int_{0 < u^1 < \cdots < u^n < t} f \circ r_j(l(u^1), \dots, l(u^n)) du^1 \cdots du^n \\ &\quad \int_{0 < v^1 < \cdots < v^m < t} g \circ r_k(l(v^1), \dots, l(v^m)) dv^1 \cdots dv^m \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \int_{\substack{0 < u^1 < \cdots < u^n < t \\ 0 < v^1 < \cdots < v^m < t}} f \circ r_j(l(u^1), \dots, l(u^n))\end{aligned}$$

$$g \circ r_k(l(v^1), \dots, l(v^m)) du^1 \dots du^n dv^1 \dots dv^m.$$

Let $w = (u^1, \dots, u^n, v^1, \dots, v^m)$. Almost surely, $u^1 < \dots < u^n, v^1 < \dots < v^m$ are different from each other. Let $s = (s^1, \dots, s^{m+n})$ be the rearrangement of w in increasing order. Almost surely, for each w , there exists a unique $\sigma \in \tilde{\mathfrak{S}}_{n,m}$ such that $s = \sigma(w)$. We change w by $\sigma^{-1}(s)$,

$$\begin{aligned} \langle l, f \rangle \langle l, g \rangle &= \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \sum_{\sigma \in \tilde{\mathfrak{S}}_{n,m} 0 < s^1 < \dots < s^{n+m} < t} \int ds^1 \dots ds^{n+m} \\ &\quad (f \circ r_j) \otimes (g \circ r_k) \circ \sigma^{-1}(l(s^1), \dots, l(s^{m+n})) \\ &= \sum_{\sigma \in \tilde{\mathfrak{S}}_{n,m} 0 < s^1 < \dots < s^{n+m} < t} \int (f \otimes g) \circ \sigma^{-1}(l(s^1), \dots, l(s^{m+n})) ds^1 \dots ds^{n+m} \\ &= \sum_{\substack{\sigma \in \tilde{\mathfrak{S}}_{n,m}^1 \\ r \in R}} \int (f \otimes g) \circ r \circ \sigma^{-1}(l(s^1), \dots, l(s^{m+n})) ds^1 \dots ds^{n+m} \\ &= \sum_{\sigma \in \tilde{\mathfrak{S}}_{n,m}^1} \langle l, (f \otimes g) \circ \sigma^{-1} \rangle \\ &= \sum_{j=0}^{m-1} \sum_{\sigma \in \tilde{\mathfrak{S}}_{n,m}} \langle l, (f \otimes (g \circ r_j)) \circ \sigma^{-1} \rangle. \end{aligned}$$

□

Corollary 3.22.

$$\mu(l^{x_1} \dots l^{x_n}) = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} V_{x_{\sigma_2}}^{x_{\sigma_1}} \dots V_{x_{\sigma_1}}^{x_{\sigma_n}}.$$

Proof.

$$l^{x_1} \dots l^{x_n} = \prod_{i=1}^n \int_0^{|l|} 1_{\{l(t_i)=x_i\}} dt_i = \int_0^{|l|} \prod_{i=1}^n 1_{\{l(t_i)=x_i\}} dt_i.$$

In the above expression, almost surely, one can write t_1, \dots, t_n in increasing order, $s_1 = t_{\sigma(1)} < \dots < s_n = t_{\sigma(n)}$ for a unique $\sigma \in \mathfrak{S}_n$. Then,

$$\begin{aligned} \int_0^{|l|} \prod_{i=1}^n 1_{\{l(t_i)=x_i\}} dt_i &= \sum_{\sigma \in \mathfrak{S}_n 0 < t_{\sigma(1)} < \dots < t_{\sigma(n)} < |l|} \int \prod_{i=1}^n 1_{\{l(t_{\sigma(i)})=x_{\sigma(i)}\}} dt_{\sigma(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n 0 < s_1 < \dots < s_n < |l|} \int \prod_{i=1}^n 1_{\{l(s_i)=x_{\sigma(i)}\}} ds_i. \end{aligned}$$

Since $\mathfrak{S}_n r_j = \mathfrak{S}_n$ for all $j = 1, \dots, n$, the above expression equals to

$$\sum_{\sigma \in \mathfrak{S}_n 0 < s_1 < \dots < s_n < |l|} \int \prod_{i=1}^n 1_{\{l(s_i)=x_{\sigma(i+j)}\}} ds_i.$$

for all $j = 1, \dots, n$. Finally,

$$\begin{aligned} l^{x_1} \dots l^{x_n} &= \frac{1}{n} \sum_{j=1}^n \sum_{\sigma \in \mathfrak{S}_n} \int_{0 < s_1 < \dots < s_n < |l|} \prod_{i=1}^n 1_{\{l(s_i) = x_{\sigma(i+j)}\}} ds_i \\ &= \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} l^{x_{\sigma(1)}, \dots, x_{\sigma(n)}}. \end{aligned}$$

Then, by Proposition 3.20, we are done. \square

Corollary 3.23. *The linear space generated by all the multi-occupation fields is an algebra.*

Proof. By shuffle product, the operation of multiplication is closed. \square

Theorem 3.24 (Blackwell's theorem, [DM78]). *Suppose (E, \mathcal{E}) is a Blackwell space, \mathcal{S}, \mathcal{F} are sub- σ -field of \mathcal{E} and \mathcal{S} is separable. Then $\mathcal{F} \subset \mathcal{S}$ iff every atom of \mathcal{F} is a union of atoms of \mathcal{S} .*

Theorem 3.25. *The family of all multi-occupation fields generates the Borel- σ -field on the loops.*

Lemma 3.26. *Suppose $(E, \mathcal{B}(E))$ is a Polish space with the Borel- σ -field. Let $\{f_i, i \in \mathbb{N}\}$ be measurable functions and denote $\mathcal{F} = \sigma(f_i, i \in \mathbb{N})$. Then, $\mathcal{F} = \mathcal{B}(E)$ iff for all $x \neq y \in E$, there exists f_i such that $f_i(x) \neq f_i(y)$.*

Proof. Since E is Polish, $\mathcal{B}(E)$ is separable and $(E, \mathcal{B}(E))$ is Blackwell space. The atoms of $\mathcal{B}(E)$ are all the one point sets. Obviously, $\mathcal{F} \subset \mathcal{B}(E)$ and \mathcal{F} is separable. By Blackwell's theorem, $\mathcal{F} = \mathcal{B}(E)$ iff the atoms of \mathcal{F} are all the one point sets which is equivalent to the following: for all $x \neq y \in E$, there exists f_i such that $f_i(x) \neq f_i(y)$. \square

Proof for Theorem 3.25. By Lemma 3.26 and the fact that

$$\{l^{x_1, \dots, x_m} : m \in \mathbb{N}_+, (x_1, \dots, x_m) \in S^m\}$$

is countable, it is sufficient to show that given all the multi-occupation fields of the loop l , the loop is uniquely determined.

Note first that the length of the loop can be recovered from the occupation field as $|l| = \sum_{x \in S} l^x$.

Let $J(l) = \max\{n \in \mathbb{N} : \exists (x_1, \dots, x_n) \in S^n \text{ such that } x_i \neq x_{i+1} \text{ for } i = 1, \dots, n-1, x_1 \neq x_n \text{ and } l^{x_1, \dots, x_n} > 0\}$, the total number of the jumps in the loop l . Define $D(l)$ to be the set of discrete pointed loop such that $l^{x_1, \dots, x_{J(l)}} > 0$. As a discrete loop is viewed as an equivalent class of discrete pointed loop, it appears that $D(l)$ is actually the discrete loop l^d . A loop is defined by the discrete loop with the corresponding holding times. It remains

to show that the corresponding holding times can be recovered from the multi-occupation field. Suppose we know that the multiplicity of the discrete loop $n(l^d) = n$, the length of the discrete loop $J(l) = qn$ and that $(x_1, \dots, x_q, \dots, x_1, \dots, x_q) \in D(l)$ is a pointed loop representing l^d . Then the loop l can be written in the following form:

$$(x_1^1, \tau_1^1, \dots, x_q^1, \tau_q^1, \dots, x_1^n, \tau_1^n, \dots, x_q^n, \tau_q^n)^o$$

with $x_j^i = x_j, i = 1, \dots, q$ and $(\tau_1^1, \dots, \tau_q^1) \geq \dots \geq (\tau_1^n, \dots, \tau_q^n)$ in the lexicographical order. For $k \in M_{n \times q}(\mathbb{N}_+)$ a n by q matrix, define $y(k) \in S^{\sum_{i,j} k_j^i}$ as follows

$$y(k) = (\underbrace{x_1^1, \dots, x_1^1}_{k_1^1 \text{ times}}, \underbrace{x_2^1, \dots, x_2^1}_{k_2^1 \text{ times}}, \dots, \underbrace{x_q^n, \dots, x_q^n}_{k_q^n \text{ times}}).$$

Define $k! = \prod_{i,j} k_j^i!$. Define $K^i = (k_1^i, \dots, k_q^i)$ for $i = 1, \dots, n$. Define $\tau^i = (\tau_1^i, \dots, \tau_q^i)$ for $i = 1, \dots, n$. For $K \in \mathbb{N}^q$ and $t \in \mathbb{R}^q$, define the polynomial $f^K(t) = \prod_{j=1}^q (t_j)^{K_j}$. We have the following expression,

$$l^{y(k)} = \frac{1}{k!} \sum_{i=1}^n (f^{K^1} \otimes \dots \otimes f^{K^n}) \circ r_i(\tau^1, \dots, \tau^n)$$

where $r_i(\tau^1, \dots, \tau^n) = (\tau^{n-i+1}, \dots, \tau^n, \tau^1, \dots, \tau^{n-i})$. All the holding times are bounded by the length $|l|$ of the loop. By the theorem of Weierstrass, for any continuous function f on $(\mathbb{R}^q)^n$, the following quantity is determined by the family of occupation fields:

$$\sum_{i=1}^n f \circ r_i(\tau^1, \dots, \tau^n).$$

As a consequence, $\sum_{i=1}^n \delta_{r_i(\tau^1, \dots, \tau^n)}$ is uniquely determined. Since we order $\tau^1 \geq \dots \geq \tau^n$ in the lexicographical order, (τ^1, \dots, τ^n) is uniquely determined. Finally, the loop l is determined by the family of the multi-occupation fields of l and we are done. \square

3.5 The occupation field in the transient case

Assumption: Throughout this section, assume we are in the transient case.

Proposition 3.27. *Suppose χ is a non-negative function on S with compact support F . Let $\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})$ be the spectral radius of $M_{\sqrt{\chi}} V M_{\sqrt{\chi}}$. Then, for $z \in D = \{z \in \mathbb{C} : \operatorname{Re}(z) < \frac{1}{\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})}\}$, the following equation holds:*

$$\mu(e^{z\langle l, \chi \rangle} - 1) = -\ln \det(I - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}}).$$

Outside of D , $\mu(|e^{z\langle l, \chi \rangle} - 1|) = \infty$.

Proof. Suppose $n = |\text{supp}(\chi)|$ and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $M_{\sqrt{\chi}} V M_{\sqrt{\chi}}$ ordered in the sense of non-increasing module. Then, $|\lambda_1| = \rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})$. By Corollary 3.22 and Proposition 3.20,

$$\mu(\langle l, \chi \rangle^m) = (m-1)! \sum_{(x^1, \dots, x^m) \in S^m} V_{y_2}^{y_1} \cdots V_{y_1}^{y_m} \chi(y_1) \cdots \chi(y_m) = (m-1)! \text{Tr}((M_{\sqrt{\chi}} V M_{\sqrt{\chi}})^m).$$

We have:

$$e^z = 1 + z + \cdots + \frac{z^n}{n!} + z^{n+1} \int_{0 < s_1 < \cdots < s_{n+1} < 1} e^{s_1 z} ds_1 \cdots ds_{n+1}.$$

Therefore

$$|e^{z+h} - 1 - z - \cdots - \frac{z^n}{n!}| \leq e^{\max(\text{Re}(z), 0)} \frac{|z|^{n+1}}{(n+1)!}.$$

In particular, $|e^x - 1| \leq e^{\max(\text{Re}(x), 0)} |x|$ and $|e^x - 1 - x| \leq e^{\max(\text{Re}(x), 0)} \frac{|x|^2}{2}$.

For $z \in \mathbb{C}$ such that $\text{Re}(z) < 1/\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}}) = 1/|\lambda_1|$, let $b = \max(\text{Re}(z), 0)$,

$$\begin{aligned} \mu(|e^{z\langle l, \chi \rangle} - 1|) &\leq \mu(e^{b\langle l, \chi \rangle} |z| \langle l, \chi \rangle) = \mu\left(\sum_{m=0}^{\infty} \frac{|z| b^m \langle l, \chi \rangle^{m+1}}{m!}\right) \\ &= \sum_{m=0}^{\infty} \frac{|z| b^m \mu(\langle l, \chi \rangle^{m+1})}{m!} = \sum_{m=0}^{\infty} |z| b^m \text{Tr}((M_{\sqrt{\chi}} V M_{\sqrt{\chi}})^{m+1}) \\ &\leq |z| |\text{supp}(\chi)| \sum_{m=0}^{\infty} b^m (\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}}))^{m+1} < \infty \end{aligned}$$

Consequently, $\Phi(z) = \mu(e^{z\langle l, \chi \rangle} - 1)$ is well-defined for $z \in D$. Next, we will show that $\Phi(z)$ is analytic in D . Fix $z_0 \in D$, take h small enough that $z_0 + h \in D$. By an argument very similar to the above one, we have that $\mu(e^{z_0\langle l, \chi \rangle} \langle l, \chi \rangle)$ and $\mu(e^{(\text{Re}(z_0) + \max(\text{Re}(h), 0))\langle l, \chi \rangle} \frac{\langle l, \chi \rangle^2}{2})$ are well-defined and finite.

$$\begin{aligned} |\Phi(z_0 + h) - \Phi(z_0) - h\mu(e^{z_0\langle l, \chi \rangle} \langle l, \chi \rangle)| \\ = \mu(|e^{z_0\langle l, \chi \rangle} (e^{h\langle l, \chi \rangle} - 1 - h\langle l, \chi \rangle)|) \\ \leq \mu\left(e^{\text{Re}(z_0)\langle l, \chi \rangle} e^{\max(\text{Re}(h), 0)\langle l, \chi \rangle} \frac{h^2 \langle l, \chi \rangle^2}{2}\right) = O(h^2). \end{aligned}$$

Finally, by dominated convergence, for $|z| < 1/\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})$,

$$\Phi(z) = \sum_{n \geq 1} \frac{1}{n} \text{Tr}((M_{\sqrt{\chi}} V M_{\sqrt{\chi}})^n) = -\ln \det(1 - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}}).$$

Since $\Phi(z)$ is analytic in $D = \{z \in \mathbb{C} : \text{Re}(z) < 1/\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})\}$, $\Phi(z)$ is the unique analytic continuation of $-\ln \det(1 - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}})$ in D .

$\ln \det(I - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}})$ cannot be defined on \mathbb{C} as an analytic function. Nevertheless, after cutting down several half lines starting from $1/\lambda_1, \dots, 1/\lambda_n$, it is analytic and equals

$-\sum_{i=1}^n \ln(1 - z\lambda_i)$. Moreover, when z converges to some λ_i , $|\ln \det(I - zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})|$ tends to infinity. But we have showed that $\mu(e^{z\langle l, \chi \rangle} - 1) = -\ln \det(I - zM_{\sqrt{\chi}}VM_{\sqrt{\chi}})$ is well-defined as an analytic function on D . Consequently, $1/\lambda_1, \dots, 1/\lambda_n$ lie in D^c , i.e. $\operatorname{Re}(\frac{1}{\lambda_i}) \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})} = \frac{1}{|\lambda_1|}$. In particular, $\lambda_1 = \rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})$. For $x \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}$,

$$\mu(|e^{x\langle l, \chi \rangle} - 1|) = \mu(e^{x\langle l, \chi \rangle} - 1) \geq \mu(e^{\frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}\langle l, \chi \rangle} - 1).$$

By monotone convergence,

$$\begin{aligned} \mu(e^{\frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}\langle l, \chi \rangle} - 1) &= \lim_{y \uparrow \lambda_1} \mu(e^{y\langle l, \chi \rangle} - 1) = \lim_{y \uparrow \lambda_1} -\ln \det(I - yM_{\chi}VM_{\chi}) \\ &= \lim_{y \uparrow \lambda_1} |-\ln \det(I - yM_{\chi}VM_{\chi})| = \infty. \end{aligned}$$

Consequently, for $x \geq \frac{1}{\rho(M_{\sqrt{\chi}}GM_{\sqrt{\chi}})}$, $\mu(|e^{x\langle l, \chi \rangle} - 1|) = \infty$. For all $y \in \mathbb{R}$, $\mu(|e^{iy\langle l, \chi \rangle} - 1|) < \infty$. Therefore, by the triangular inequality, for $z = x + iy \notin D$,

$$\begin{aligned} \mu(|e^{z\langle l, \chi \rangle} - 1|) &\geq |\mu(|e^{z\langle l, \chi \rangle} - e^{iy\langle l, \chi \rangle}|) - \mu(|e^{iy\langle l, \chi \rangle} - 1|)| \\ &= |\mu(|e^{x\langle l, \chi \rangle} - 1|) - \mu(|e^{iy\langle l, \chi \rangle} - 1|)| = \infty. \end{aligned}$$

□

Lemma 3.28. *Suppose χ is a finitely supported non-negative function on S and F contains the support of χ . Then,*

$$\begin{aligned} \frac{\det(V_F)}{\det((V_{\chi})_F)} &= \det(I + (M_{\chi})_F V_F) = \det(I + M_{\sqrt{\chi}}VM_{\sqrt{\chi}}) \\ &= \begin{vmatrix} I_F & V_F \\ -(M_{\chi})_F & I_F \end{vmatrix} = 1 + \sum_{A \subset F, A \neq \emptyset} \prod_{x \in A} \chi(x) V_A. \end{aligned}$$

Proof. By the resolvent equation, we have $V_F = (V_F)_{\chi} + (V_F)_{\chi}(M_{\chi})_F V_F$. By Proposition 2.7, we have $(V_{\chi})_F = (V_F)_{\chi}$. Combining these two results, we have $V_F = (V_{\chi})_F + (V_{\chi})_F(M_{\chi})_F V_F$. Consequently,

$$\frac{\det(V_F)}{\det((V_{\chi})_F)} = \det(I + (M_{\chi})_F V_F).$$

The last equality follows from simple calculations in linear algebra. □

Corollary 3.29. *For non-negative χ not necessarily finitely supported,*

$$e^{\mu(1 - e^{-\langle l, \chi \rangle})} = 1 + \sum_{F \subset S, 0 < |F| < \infty} \prod_{x \in F} \chi(x) \det(V_F)$$

Proof. For χ a non-negative finitely supported function, by Proposition 3.27 with Lemma 3.28,

$$\begin{aligned} e^{\mu(1-e^{-\langle l, \chi \rangle})} &= \det(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}}) = \det(I + (M_{\chi})_{\text{supp}(\chi)} V_{\text{supp}(\chi)}) \\ &= 1 + \sum_{F \subset \text{supp}(\chi), F \neq \emptyset} \left(\prod_{x \in F} \chi(x) \right) \det(V_F). \end{aligned}$$

The trace of the Markov process on F has the potential V_F and generator \tilde{L} . Since $\det(-\tilde{L}) > 0$ and $(-\tilde{L})V_F = Id$, $\det(V_F) > 0$. Finally, the result comes from monotone convergence theorem. \square

Corollary 3.30. *For $a \geq 0$, let $\chi = a\delta_x$, then $\mu(1 - e^{-aV_x^x}) = \ln(1 + aV_x^x)$. As a result, $\mu(l^x \in dt) = \frac{1}{t} e^{-t/V_x^x} dt$ for $t > 0$.*

Proposition 3.31. *For non-negative function χ ,*

$$\begin{aligned} \mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l, \chi \rangle})) &= \ln\left(\prod_{x \in S} \frac{\chi(x) - L_x^x}{-L_x^x}\right) \\ \mu(1_{\{l \text{ is non-trivial}\}}(1 - e^{-\langle l, \chi \rangle})) &= \ln(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}}) + \ln\left(\prod_{x \in S} \frac{-L_x^x}{\chi(x) - L_x^x}\right). \end{aligned}$$

Proof. Since $\mu(p(\xi) = 1, \xi_1 = x, \tau_1 \in dt^1) = \frac{e^{L_x^x t^1}}{t^1} dt^1$,

$$\mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l, \chi \rangle})) = \sum_{x \in S} \int_0^\infty \frac{e^{L_x^x t^1}}{t^1} (1 - e^{-\chi(x)t^1}) dt^1 = \ln\left(\prod_{x \in S} \frac{\chi(x) - L_x^x}{-L_x^x}\right).$$

Combining with Proposition 3.27, we have

$$\begin{aligned} \mu(1_{\{l \text{ is non-trivial}\}}(1 - e^{-\langle l, \chi \rangle})) &= \mu(1 - e^{-\langle l, \chi \rangle}) - \mu(1_{\{l \text{ is trivial}\}}(1 - e^{-\langle l, \chi \rangle})) \\ &= \ln(\det(I + M_{\sqrt{\chi}} V M_{\sqrt{\chi}})) + \ln\left(\prod_{x \in S} \frac{-L_x^x}{\chi(x) - L_x^x}\right). \end{aligned}$$

\square

Proposition 3.32. *If χ_1, \dots, χ_n are finitely supported non-negative functions on S , and for A a subset of $\{1, \dots, n\}$ we set $\chi_A = \sum_{i \in A} \chi_i$, then for $n \geq 2$,*

$$\begin{aligned} \mu\left(\prod_{i=1}^n (1 - e^{-\langle l, \chi_i \rangle})\right) &= - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \ln \det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}}); \\ \mu(1_{\{l \text{ is trivial}\}} \prod_{i=1}^n (1 - e^{-\langle l, \chi_i \rangle})) &= - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \ln\left(\prod_{x \in F_A} \frac{-L_x^x + \chi_A(x)}{-L_x^x}\right). \end{aligned}$$

Proof. We see that

$$\prod_{i=1}^n (1 - e^{-\langle l, \chi_i \rangle}) = - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} (1 - e^{-\langle l, \chi_A \rangle}).$$

Therefore,

$$\begin{aligned} \mu\left(\prod_{i=1}^n (1 - e^{-\langle l, \chi_i \rangle})\right) &= - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \mu(1 - e^{-\langle l, \chi_A \rangle}) \\ &= - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \ln \det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}}). \end{aligned}$$

The last equality is deduced from Proposition 3.27. By a similar method and Proposition 3.31, we get the following expression for the trivial loops:

$$\mu(1_{\{l \text{ is trivial}\}} \prod_{i=1}^n (1 - e^{-\langle l, \chi_i \rangle})) = - \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \ln\left(\prod_{x \in F_A} \frac{-L_x^x + \chi_A(x)}{-L_x^x}\right).$$

□

Proposition 3.33. *For a finite subset $F \subset S$,*

$$\mu(l \text{ is non-trivial and } l \text{ visits } F) = \ln\left(\prod_{x \in F} (-L_x^x) \det(V_F)\right).$$

Proof. By Proposition 3.31,

$$\mu(1_{\{l \text{ is non-trivial}\}} (1 - e^{-\langle l, t1_F \rangle})) = \ln \det(I + M_{\sqrt{t1_F}} V M_{\sqrt{t1_F}}) + \ln\left(\prod_{x \in F} \frac{-L_x^x}{t - L_x^x}\right).$$

By Lemma 3.28, we have

$$\ln \det(I + M_{\sqrt{t1_F}} V M_{\sqrt{t1_F}}) = \ln\left(1 + \sum_{A \subset F, A \neq \emptyset} t^{|A|} V_A\right).$$

Take $t \rightarrow \infty$, we have

$$\mu(1_{\{l \text{ is non-trivial and } l \text{ visits } F\}}) = \ln\left(\prod_{x \in F} (-L_x^x) \det(V_F)\right).$$

□

Similarly, one has the following property.

Proposition 3.34. *Suppose we are given $n \geq 2$ finite subset F_1, \dots, F_n . For any subset $A \subset \{1, \dots, n\}$, define $F_A = \bigcup_{i \in A} F_i$. Then,*

$$\begin{aligned} \mu(l \text{ is not trivial and it visits all } F_i \text{ for } i = 1, \dots, n) \\ = - \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \det(V_{F_A}) + \sum_{x \in \bigcap_{i=1}^n F_i} \ln(-L_x^x). \end{aligned}$$

Proof. By Proposition 3.32, take $\chi_i = t1_{F_i}$:

$$\begin{aligned} \mu(1_{\{l \text{ is non-trivial}\}} \prod_{i=1}^n (1 - e^{-\langle l, t1_{F_i} \rangle})) \\ = - \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}}) \\ + \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \left(\prod_{x \in F_A} \frac{-L_x^x + \chi_A(x)}{-L_x^x} \right). \end{aligned}$$

where $\chi_A = \sum_{i \in A} t1_{F_i}$. By Lemma 3.28, for A non-empty,

$$\begin{aligned} \det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}}) &= 1 + \sum_{B \subset F_A, B \neq \emptyset} t^{|B|} \left(\prod_{x \in B} \left(\sum_{i \in A} 1_{\{x \in F_i\}} \right) \right) \det(V_B) \\ &\sim t^{|F_A|} \left(\prod_{x \in F_A} \left(\sum_{i \in A} 1_{\{x \in F_i\}} \right) \right) \det(V_{F_A}) \text{ as } t \rightarrow \infty. \end{aligned}$$

And we have that

$$\prod_{x \in F_A} \frac{-L_x^x + \chi_A(x)}{-L_x^x} \sim t^{|F_A|} \prod_{x \in F_A} \frac{\sum_{i \in A} 1_{\{x \in F_i\}}}{L_x^x} \text{ as } t \rightarrow \infty.$$

As a result,

$$\begin{aligned} \lim_{t \rightarrow \infty} (-1)^A \left(\ln \det(I + M_{\sqrt{\chi_A}} V M_{\sqrt{\chi_A}}) + \ln \left(\prod_{x \in F_A} \frac{-L_x^x + \chi_A(x)}{-L_x^x} \right) \right) \\ = - \ln \det(V_{F_A}) - \ln \left(\prod_{x \in F_A} (-L_x^x) \right). \end{aligned}$$

Then,

$$\begin{aligned} \mu(l \text{ is not trivial and it visits all } F_i \text{ for } i = 1, \dots, n) \\ = \lim_{t \rightarrow \infty} \mu(1_{\{l \text{ is non-trivial}\}} \prod_{i=1}^n (1 - e^{-\langle l, t1_{F_i} \rangle})) \\ = - \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \det(V_{F_A}) - \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \left(\prod_{x \in F_A} (-L_x^x) \right). \end{aligned}$$

Finally, by inclusion-exclusion principle, we have

$$\begin{aligned} \mu(l \text{ is not trivial and it visits all } F_i \text{ for } i = 1, \dots, n) \\ = - \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} (-1)^{|A|} \ln \det(V_{F_A}) + \sum_{x \in \bigcap_{i=1}^n F_i} \ln(-L_x^x). \end{aligned}$$

□

Corollary 3.35. For $n \geq 2$ and n different states x_1, \dots, x_n ,

$$\mu(l \text{ visits each state of } \{x_1, \dots, x_n\}) = - \sum_{A \subset \{x_1, \dots, x_n\}, A \neq \emptyset} (-1)^{|A|} \ln \det(V_A).$$

Definition 3.20. For a loop l , let $N(l)$ be the number of different points visited by the loop. That is $N(l) = \sum_{x \in S} 1_{\{l^x > 0\}}$.

Corollary 3.36.

$$\mu(N 1_{\{N > 1\}}) = \sum_{x \in S} \ln(-L_x^x V_x^x).$$

Proof.

$$\mu(N 1_{\{N > 1\}}) = \sum_{x \in S} \mu(l \text{ is non-trivial and } l \text{ visits } x).$$

□

Corollary 3.37.

$$\mu(N^2 1_{\{N > 1\}}) = \sum_{x, y \in S; x \neq y} \ln\left(\frac{V_x^x V_y^y}{V_x^x V_y^y - V_y^x V_x^y}\right) + \sum_{x \in S} \ln(-L_x^x V_x^x).$$

Proof.

$$\mu(N^2 1_{\{N > 1\}}) = \sum_{x, y} \mu(l \text{ is non-trivial, } l \text{ visits } x \text{ and } l \text{ visits } y).$$

□

Consider the Laguerre-type polynomial L_k with generating function

$$e^{\frac{ut}{1+t}} - 1 = \sum_{k=1}^{\infty} t^k L_k(u).$$

Lemma 3.38.

$$\sum_{k=1}^{\infty} |t^k L_k(u)| \leq e^{\frac{|ut|}{1-|t|}} - 1.$$

Proof.

$$\sum_{k=1}^{\infty} t^k L_k(u) = e^{\frac{ut}{1+t}} - 1 = \sum_{k=1}^{\infty} \left(\frac{ut}{1+t}\right)^k = \sum_{k=1}^{\infty} u^k t^k (1-t+t^2-\dots)^k.$$

Therefore,

$$\sum_{k=1}^{\infty} |t^k L_k(u)| \leq \sum_{k=1}^{\infty} |u|^k |t|^k (1+|t|+|t|^2+\dots)^k = e^{\frac{|ut|}{1-|t|}} - 1.$$

□

Proposition 3.39. $(\sqrt{k}(V_x^x)^{k/2} L_k\left(\frac{l^x}{V_x^x}\right), k \geq 1)$ are orthonormal in $L^2(\mu)$. More generally,

$$\mu\left(\sqrt{j}(V_x^x)^{j/2} L_j\left(\frac{l^x}{V_x^x}\right) \sqrt{k}(V_y^y)^{k/2} L_k\left(\frac{l^y}{V_y^y}\right)\right) = \delta_k^j.$$

Proof. $\forall s, t \leq 0$ with $|s|, |t|$ small enough with $\frac{V_x^x s}{1-sV_x^x}, \frac{V_y^y t}{1-tV_y^y} < 1/2$

$$\begin{aligned} \mu \left(\left(\sum_1^\infty (V_x^x s)^k L_k \left(\frac{l^x}{V_x^x} \right) \right) \left(\sum_1^\infty (V_y^y t)^k L_k \left(\frac{l^y}{V_y^y} \right) \right) \right) &= \mu \left((e^{\frac{l^x s}{1+V_x^x s}} - 1)(e^{\frac{l^y t}{1+V_y^y t}} - 1) \right) \\ &= \mu \left(1 - e^{\frac{l^x s}{1+V_x^x s}} \right) + \mu \left(1 - e^{\frac{l^y t}{1+V_y^y t}} \right) - \mu \left(1 - e^{\frac{l^x s}{1+V_x^x s} + \frac{l^y t}{1+V_y^y t}} \right) = -\ln(1 - stV_y^x V_x^y). \end{aligned}$$

Recall that $\mu((l^x)^n) = (n-1)!(V_x^x)^n$. By Lemma 3.38,

$$\begin{aligned} \mu \left(\left(\sum_{k=1}^\infty |(V_x^x s)^k L_k(l^x/V_x^x)|^2 \right) \right) &\leq \mu \left(\left(e^{\frac{l^x s}{1+V_x^x s}} - 1 \right)^2 \right) = \sum_{n,m=1}^\infty \frac{1}{n!} \frac{1}{m!} \mu \left(\left(\frac{l^x s}{1+V_x^x s} \right)^{n+m} \right) \\ &= \sum_{n,m=1}^\infty \frac{1}{n+m} \frac{(n+m)!}{n!m!} \left(\frac{V_x^x s}{1+V_x^x s} \right)^{n+m} \\ &\leq \sum_{k \geq 1} \left(\frac{2V_x^x s}{1+V_x^x s} \right)^k < \infty. \end{aligned}$$

Therefore, $(\frac{1}{\sqrt{k}}(V_x^x)^k L_k(\frac{l^x}{V_x^x}), k \geq 1) \in L^2(\mu)$. Moreover, in the equation

$$\mu \left(\left(\sum_1^\infty (V_x^x s)^k L_k(l^x/V_x^x) \right) \left(\sum_1^\infty (V_y^y t)^k L_k(l^y/V_y^y) \right) \right) = -\ln(1 - stV_y^x V_x^y).$$

we can expand both sides as series of s and t , compare the coefficients and deduce that

$$\mu \left((V_x^x)^j L_j \left(\frac{l^x}{V_x^x} \right) (V_y^y)^k L_k \left(\frac{l^y}{V_y^y} \right) \right) = \delta_k^j (V_y^x V_x^y)^k / k.$$

Therefore,

$$\mu \left(\sqrt{j} (V_x^x)^{j/2} L_j \left(\frac{l^x}{V_x^x} \right) \sqrt{k} (V_y^y)^{k/2} L_k \left(\frac{l^y}{V_y^y} \right) \right) = \delta_k^j.$$

□

3.6 The recurrent case

Proposition 3.40.

$$\mu(l^x \in ds, l^x > 0) = \frac{1}{s} ds.$$

Proof.

$$\mu(l^x e^{-pl^x}) = \mu(L - M_{p\delta_x}, l^x) = (V_{p\delta_x})_x^x = 1/p.$$

Therefore, $\mu(l^x \in ds, l^x > 0) = \frac{1}{s} 1_{\{s>0\}} ds$. □

Lemma 3.41. *In the irreducible positive-recurrent case, there is a one-to-one correspondence between the semi-group of the Markov process and the Markovian loop measure.*

Proof. It is enough to show the loop measure determines the law of the Markov process. Let π be the invariant probability of the Markov process. Then it is positive everywhere. Define a based loop functional $\phi_{t,x,y}^b$ as follows: for any based loop l with length $|l|$, extend the function $(l(t), t \in [0, |l|])$ periodically, i.e. by setting $l(s + |l|) = l(s)$, and set:

$$\phi_{t,x,y}^b(l) = 1_{\{|l|>t\}} \int_0^{|l|} 1_{\{l(s)=x\}} 1_{\{l(s+t)=y\}} ds.$$

This rotation invariant functional defines a loop functional $\phi_{t,x,y}$ on the space of loops.

$$\frac{\mu(|l| \in du, \phi_{t,x,y})}{du} = \frac{1}{u} \sum_{z \in S} \mathbb{P}_{u,z}^z \left[\int_0^{|l|} 1_{\{l(s)=x\}} 1_{\{l(s+t)=y\}} ds \right] = (P_t)_y^x (P_{u-t})_x^y.$$

Taking u tends to infinity,

$$\lim_{u \rightarrow \infty} \frac{\mu(|l| \in du, \phi_{t,x,y})}{du} = \pi_x (P_t)_y^x.$$

Since $\mu(l^x e^{-p|l|}) = \mu(L - p, l^x) = (V_p)_x^x$, $\pi_x = \lim_{p \rightarrow 0} p \mu(l^x e^{-p|l|})$. Finally, we are able to determine the semi-group $(P_t)_y^x$ for all x, t, y . Accordingly, the law of the Markov process is uniquely determined. \square

Remark 6. From the argument above, we see that an irreducible positive-recurrent semi-group cannot have the same loop measure as another irreducible transient or null-recurrent semi-group.

Finally, we can prove the following theorem.

Theorem 3.42. *In the irreducible recurrent case, there is a 1-1 correspondence between the semi-group of the Markov process and the Markovian loop measure.*

Proof. Given a minimal semi-group $(P_t, t \geq 0)$, we can always define the corresponding Markovian loop measure. It is left to show that we can recover the semi-group from the loop measure. Let the series of finite subset $F_1 \subset F_2 \subset \dots \subset F_n \subset \dots$ exhaust S . By Proposition 3.9, we know that the measure of the trace of the Markovian loop on F_i corresponds to the trace the Markov process on F_n . Since $|F_i| < \infty$, the trace of the Markov process on F_i is an irreducible and positive-recurrent Markov process. Let $(P_t^{(n)}, t \geq 0)$ be its semi-group. By Lemma 3.41, we can conclude that this trace of the Markov process is determined by the Markovian loop measure. Recall that $Y_t^{(n)}, t \geq 0$, the trace of the Markov process $X_t, t \geq 0$ on F_n is defined as follows:

$$A_t^{(n)} = \int_0^t 1_{\{X_s \in F_n\}} ds, \sigma_t^{(n)} \text{ is the right-continuous inverse of } A_t^{(n)}, t \geq 0 \text{ and } Y_t^{(n)} = X_{\sigma_t^{(n)}}^{(n)}, t \geq 0.$$

As n tends to infinity, $A_t^{(n)}$ increases to t and $\sigma_t^{(n)}$ decreases to t . Since $X_t, t \geq 0$ is right-continuous, $\lim_{n \rightarrow \infty} Y_t^{(n)} = X_t$. As a consequence, for any bounded f , $P_t f(x) = \lim_{n \rightarrow \infty} P_t^{(n)} f(x)$. Thus, we recover the semi-group P_t as the limit. \square

4 Poisson process of loops

In this section, we study the Poisson point processes naturally defined on the set of Markov loops (which are also known as “loop soups”). We mostly focus on the associated occupation fields and on the partitions defined by loop clusters.

4.1 Definitions and some basic properties

Definition 4.1. We denote by \mathcal{L} the Poisson point process on $\mathbb{R}_+ \times \text{loops}$ with intensity Lebesgue $\otimes \mu$ and by \mathcal{L}_α the Poisson random measure on the space of loops, $\mathcal{L}_\alpha(B) = \mathcal{L}([0, \alpha] \times B)$. Its intensity is $\alpha\mu$.

The following proposition is taken from [Kin93].

Proposition 4.1. *Let \mathcal{P} be a Poisson random measure on S with σ -finite intensity measure $\mu(dl)$.*

a) Suppose that Φ is a measurable complex valued function, with $\mu(|\text{Im}(\Phi)| \wedge 1) < \infty$ and $\mu(|e^\Phi - 1|) < \infty$, then

$$\mathbb{E}[\exp(\sum_{l \in \mathcal{P}} \Phi(l))] = e^{\alpha \int (e^{\Phi(l)} - 1) \mu(dl)}$$

b) The above equation holds if Φ is non-negative measurable without further assumptions.

c) Suppose F_1, \dots, F_k are non-negative functions, then the following ‘Campbell formula’ holds

$$\mathbb{E}[\sum_{l_1, \dots, l_k \in \mathcal{P} \text{ distinct}} \prod_{i=1}^k F_i(l_i)] = \prod_{i=1}^k \mu(F_i)$$

d) Suppose that S, T are two measurable spaces and $\phi : S \rightarrow T$ is a measurable mapping. Let \mathcal{P} be a Poisson random measure on S with intensity μ . Then $\phi \circ \mathcal{P}$ is the Poisson random measure on T with intensity $\phi \circ \mu$.

Proof. See [Kin93]. \square

From the expression of μ on trivial loops, we get the following:

Proposition 4.2. *Let $\mathcal{L}_{\alpha, \text{Trivial}, x} = \{l \in \mathcal{L}_\alpha : l \text{ is a trivial loop at } x\}$. Then, $\{|l| : l \in \mathcal{L}_{\alpha, \text{Trivial}, x}\}$ is a Poisson point measure on \mathbb{R}_+ with intensity $\frac{\alpha}{t} e^{L_x^t} dt$.*

Recall that a Poisson-Dirichlet distribution has a representation by a Poisson point process, see section 9.4 in [Kin93].

Corollary 4.3.

$$\left\{ \frac{\ell^x}{\sum_{l \in \mathcal{L}_{\alpha, \text{Trivial}, x}} l^x}; \ell \in \mathcal{L}_{\alpha, \text{Trivial}, x} \right\}$$

follows a Poisson-Dirichlet $(0, \alpha)$ distribution. Moreover, it is independent of $\sum_{l \in \mathcal{L}_{\alpha, \text{Trivial}, x}} l^x$ which follows the $\Gamma(\alpha, (-L_x^x)^{-1})$ distribution.

Recall that the density of $\Gamma(\alpha, \beta)$ distribution is $\frac{\beta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$.

Proof. By Proposition 4.1,

$$\begin{aligned} \mathbb{E}[\exp(-\lambda \sum_{l \in \mathcal{L}_{\alpha, \text{Trivial}, x}} l^x)] &= \exp\left(\int_0^\infty (e^{-\lambda t} - 1) \frac{\alpha}{t} e^{L_x^x t} dt\right) \\ &= \exp\left(\int_0^\infty \int_0^\infty \alpha (e^{-\lambda t} - 1) e^{-ts} e^{L_x^x t} ds dt\right) \\ &= \exp\left(\int_0^\infty \frac{\alpha}{\lambda + s - L_x^x} - \frac{\alpha}{s - L_x^x} ds\right) = \left(\frac{-L_x^x}{\lambda - L_x^x}\right)^\alpha \end{aligned}$$

which is exactly the Laplace transform of the $\Gamma(\alpha, (-L_x^x)^{-1})$ distribution. Therefore,

$\sum_{l \in \mathcal{L}_{\alpha, \text{Trivial}, x}} l^x$ follows the $\Gamma(\alpha, (-L_x^x)^{-1})$ distribution. \square

By taking the trace of the loops on $\{x\}$, we get a Poisson ensemble of Markov loops. To be more precise, we get a Poisson ensemble of trivial loops at x , but its intensity measure, (i.e. the loop measure), is associated with the generator $(L_{\{x\}})_x^x = -1/V_x^x$. As a consequence, we have the following proposition:

Proposition 4.4. $\hat{\mathcal{L}}_\alpha^x = \sum_{l \in \mathcal{L}_\alpha} l^x$ follows a $\Gamma(\alpha, V_x^x)$ distribution. $\{\frac{l^x}{\hat{\mathcal{L}}_\alpha^x}, l \in \mathcal{L}_\alpha\}$ follows a Poisson-Dirichlet distribution $\Gamma(0, \alpha)$ which is independent of $\hat{\mathcal{L}}_\alpha^x$.

Definition 4.2. Define $\hat{\mathcal{L}}_\alpha^x = \sum_{l \in \mathcal{L}_\alpha} l^x$ and $\langle \hat{\mathcal{L}}_\alpha, \chi \rangle = \sum_{x \in S} \hat{\mathcal{L}}_\alpha^x \chi(x)$.

Proposition 4.5. For any non-negative measurable χ on S ,

$$\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle}] = (1 + \sum_{A \subset S, 0 < |A| < \infty} \prod_{x \in A} \chi(x) \det(V_A))^{-\alpha}.$$

For any non-negative finitely supported χ on S and $z \in D = \{z \in \mathbb{C} : \text{Re}(z) < \frac{1}{\rho(M_{\sqrt{\chi}} V M_{\sqrt{\chi}})}\}$,

$$\mathbb{E}[e^{z \langle \hat{\mathcal{L}}_\alpha, \chi \rangle}] = (\det(I - z M_{\sqrt{\chi}} V M_{\sqrt{\chi}}))^{-\alpha}.$$

Outside of D , $\mathbb{E}[|e^{z \langle \hat{\mathcal{L}}_\alpha, \chi \rangle}|] = \infty$.

Proof. It is a direct consequence of Proposition 3.27, Corollary 3.29 and Proposition 4.1. \square

Proposition 4.6. $\hat{\mathcal{L}}_1^x$ is exponentially distributed with parameter $1/V_x^x$.

Proof. Since $\mathbb{E}[e^{-p\hat{\mathcal{L}}_1^x}] = \frac{1}{1+pV_x^x}$ and $\hat{\mathcal{L}}_1^x \geq 0$, $\hat{\mathcal{L}}_1^x$ is exponentially distributed with parameter $1/V_x^x$. \square

Remark 7.

$$\mathbb{E}((1 - e^{-\frac{\hat{\mathcal{L}}_1^x}{V_x^x}})^{-1}) = \zeta(\alpha), \alpha > 1.$$

Proof. By Proposition 4.5

$$\mathbb{E}((1 - e^{-\frac{\hat{\mathcal{L}}_1^x}{V_x^x}})^{-1}) = \sum_{k=0}^{\infty} \mathbb{E}(e^{-\frac{k}{V_x^x} \hat{\mathcal{L}}_1^x}) = \sum_{k=1}^{\infty} k^{-\alpha} = \zeta(\alpha).$$

\square

4.2 Moments and polynomials of the occupation field

Definition 4.3 (α -permanent). Denote by $m(\sigma)$ the number of cycles in the decomposition of the permutation σ . For any square matrix $A = (A_j^i, i, j = 1, \dots, n)$, define the α -permanent of A as

$$\text{Per}_{\alpha}(A) = \sum_{\sigma \in S_n} \alpha^{m(\sigma)} A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

Note that $\text{Per}_{-1}(A) = \det(-A)$.

Proposition 4.7.

$$\mathbb{E}[\hat{\mathcal{L}}_{\alpha}^{x_1} \cdots \hat{\mathcal{L}}_{\alpha}^{x_n}] = \text{Per}_{\alpha}((V_{x_m}^{x_l})_{1 \leq m, l \leq n}).$$

Proof. Let $F_i(l) = l^{x_i}$. By Corollary 3.22,

$$\mu(F_1 \cdots F_k) = \mu(l^{x_1} \cdots l^{x_n}) = \frac{1}{k} \sum_{\sigma \in \mathfrak{S}_k} V_{x_{\sigma(2)}}^{x_{\sigma(1)}} \cdots V_{x_{\sigma(k)}}^{x_{\sigma(1)}} = \sum_{\sigma \in \mathfrak{S}_k, m(\sigma)=1} V_{x_{\sigma(1)}}^{x_1} \cdots V_{x_{\sigma(k)}}^{x_k}.$$

Let $\mathcal{P}(\{1, \dots, n\})$ be the collection of partitions of $\{1, \dots, n\}$. For a partition π , we denote by $\#\pi$ the number of blocks in π , $\pi = (\pi_1, \dots, \pi_{\#\pi})$.

$$\begin{aligned} \mathbb{E}[\hat{\mathcal{L}}_{\alpha}^{x_1} \cdots \hat{\mathcal{L}}_{\alpha}^{x_n}] &= \mathbb{E}[(\sum_{l \in \mathcal{L}_{\alpha}} l^{x_1}) \cdots (\sum_{l \in \mathcal{L}_{\alpha}} l^{x_n})] \\ &= \sum_{\pi \in \mathcal{P}(\{1, \dots, n\})} \mathbb{E}[\sum_{\substack{l_1, \dots, l_{\#\pi} \in \mathcal{L}_{\alpha} \\ \text{distinct}}} \prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i)]. \end{aligned}$$

Define $G_i^\pi = \prod_{j \in \pi_i} F_j$ for $j = 1, \dots, \#\pi$. Then,

$$\prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i) = \prod_{i=1}^{\#\pi} G_i(l_i).$$

By Campbell's formula,

$$\mathbb{E} \left[\sum_{\substack{l_1, \dots, l_{\#\pi} \in \mathcal{L}_\alpha \\ \text{distinct}}} \prod_{i=1}^{\#\pi} (\prod_{j \in \pi_i} F_j)(l_i) \right] = \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \mu(G_i).$$

Write π_i in decreasing order $p(i, 1) < \dots < p(i, \#\pi_i)$, then

$$\mu(G_i) = \sum_{\sigma \in \mathfrak{S}_{\#\pi_i}, m(\sigma)=1} \prod_{j=1}^{\#\pi_i} V_{x_{p(i, \sigma(j))}}^{x_{p(i, j)}} = \sum_{\substack{\sigma: \text{circular} \\ \text{permutation} \\ \text{on } \pi_i}} \prod_{j \in \pi_i} V_{x_{\sigma(j)}}^{x_j}.$$

Clearly, there is a one-to-one correspondence between a permutation η on $\{1, \dots, n\}$ and an $m(\eta)$ -partition $\pi = (\pi_1, \dots, \pi_{m(\eta)})$ together with these circular permutation on the blocks of π . Finally,

$$\begin{aligned} \mathbb{E}[\hat{\mathcal{L}}_\alpha^{x_1} \dots \hat{\mathcal{L}}_\alpha^{x_n}] &= \sum_{\pi \in \mathcal{P}(\{1, \dots, n\})} \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \mu(G_i) \\ &= \sum_{\pi \in \mathcal{P}(\{1, \dots, n\})} \alpha^{\#\pi} \prod_{i=1}^{\#\pi} \sum_{\substack{\sigma: \text{circular} \\ \text{permutation} \\ \text{on } \pi_i}} \prod_{j \in \pi_i} V_{x_{\sigma(j)}}^{x_j} \\ &= \sum_{\eta \in \mathfrak{S}_n} \alpha^{m(\eta)} \prod_{i=1}^n V_{x_{\eta(i)}}^{x_i} = \text{Per}_\alpha(V_{x_j}^{x_i}, 1 \leq i, j \leq n) \end{aligned}$$

□

Definition 4.4. $\mu(l^x) = V_x^x$, define $\tilde{\mathcal{L}}_\alpha^x = \hat{\mathcal{L}}_\alpha^x - \alpha V_x^x$.

Note that $\mathbb{E}[\tilde{\mathcal{L}}_\alpha^x] = 0$.

Definition 4.5. For $A = (A_{ij})_{1 \leq i, j \leq n}$, define

$$\text{Per}_\alpha^0(A) = \sum_{\sigma \in S_n, \sigma(i) \neq i, i=1, \dots, n} \alpha^{m(\sigma)} A_{1\sigma(1)} \dots A_{n\sigma(n)}$$

with $m(\sigma)$ the number of cycles in σ .

Proposition 4.8.

$$\mathbb{E}[\tilde{\mathcal{L}}_\alpha^{x_1} \dots \tilde{\mathcal{L}}_\alpha^{x_n}] = \text{Per}_\alpha^0((V_{x_m}^{x_l})_{1 \leq m, l \leq n}).$$

Proof. For $\sigma \in S_n$, let $n(k, \sigma)$ be the number of cycles of length k in σ . According to Proposition 4.7,

$$\begin{aligned} \mathbb{E}[\tilde{\mathcal{L}}_\alpha^{x_1} \cdots \tilde{\mathcal{L}}_\alpha^{x_n}] &= \mathbb{E}\left[\prod_{i=1}^n (\hat{\mathcal{L}}_\alpha^{x_i} - \alpha V_{x_i}^{x_i})\right] = \sum_{A \subset \{1, \dots, n\}} (-\alpha)^{|A|} \left(\prod_{j \in A} V_{x_j}^{x_j}\right) \mathbb{E}\left[\prod_{j \in A^c} \hat{\mathcal{L}}_\alpha^{x_j}\right] \\ &= \sum_{A \subset \{1, \dots, n\}} (-\alpha)^{|A|} \left(\prod_{j \in A} V_{x_j}^{x_j}\right) \text{Per}_\alpha(V_{A^c}) \end{aligned}$$

where $A^c = \{1, \dots, n\} \setminus A$. The above quantity equals

$$\begin{aligned} \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} \sum_{\sigma \in \mathfrak{S}_n, \sigma|_A = \text{Id}} \alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1} \cdots V_{x_{\sigma(n)}}^{x_n} \\ = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1} \cdots V_{x_{\sigma(n)}}^{x_n} \left(\sum_{A \subset \{1, \dots, n\}} 1_{\{\sigma|_A = \text{Id}\}} (-1)^{|A|} \right) \\ = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{m(\sigma)} V_{x_{\sigma(1)}}^{x_1} \cdots V_{x_{\sigma(n)}}^{x_n} 1_{\{n(1, \sigma) = 0\}} = \text{Per}_\alpha^0((V_{x_m}^{x_l})_{1 \leq m, l \leq n}). \end{aligned}$$

□

It is well-known that the generalized Laguerre polynomials $(L_k^{\alpha-1}, k \in \mathbb{N}), \alpha > 0$ have the following generating function

$$\frac{e^{-\frac{xt}{1-t}}}{(1-t)^\alpha} = \sum_{k=0}^{\infty} t^k L_k^{\alpha-1}(x).$$

Moreover,

$$L_k^\alpha(x) = \sum_{i=0}^k (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}.$$

Definition 4.6. Define $P_k^{\alpha, \sigma}(x) = (-\sigma)^k L_k^{\alpha-1}\left(\frac{x}{\sigma}\right)$ and $Q_k^{\alpha, \sigma}(x) = P_k^{\alpha, \sigma}(x + \alpha\sigma)$.

These polynomials of the occupation field are related to Wick renormalisation in the symmetric case, when α is a half integer (see [LJ11]).

Proposition 4.9.

- a) $\frac{e^{\frac{xt}{1+t\sigma}}}{(1+t\sigma)^\alpha} = \sum_{k=0}^{\infty} t^k P_k^{\alpha-1, \sigma}(x)$ and $\frac{e^{\frac{xt+t\alpha\sigma}{1+t\sigma}}}{(1+t\sigma)^\alpha} = \sum_{k=0}^{\infty} t^k Q_k^{\alpha-1, \sigma}(x)$.
- b) $P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x) = Q_k^{\alpha, V_x^x}(\tilde{\mathcal{L}}_\alpha^x)$.
- c) $\mathbb{E}[P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x) P_l^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x)] = \delta_l^k \binom{\alpha+k}{k} (V_y^x V_x^y)^k$.

Proof of c). By Proposition 4.5, for $|s|$ and $|t|$ small enough,

$$\mathbb{E}\left[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+tV_x^x}}}{(1+tV_x^x)^\alpha} \frac{e^{\frac{s\hat{\mathcal{L}}_\alpha^y}{1+sV_y^y}}}{(1+sV_y^y)^\alpha}\right] = (1 - stV_y^x V_x^y)^{-\alpha}.$$

Since $L_k^\alpha(x) = \sum_{i=0}^k (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}$, $|L_k^\alpha(x)| \leq \sum_{i=0}^k \binom{n+\alpha}{n-i} \frac{|x|^i}{i!} = L_k^\alpha(-|x|)$. Therefore,

$$\begin{aligned} \sum_{k,l \in \mathbb{N}} |t^k P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x) s^l P_l^{\alpha, V_y^y}(\hat{\mathcal{L}}_\alpha^y)| &\leq \sum_{k,l \in \mathbb{N}} (|t| V_x^x)^k L_k^{\alpha-1}(-\frac{\hat{\mathcal{L}}_\alpha^x}{V_x^x}) (|s| V_y^y)^l L_l^{\alpha-1}(-\frac{\hat{\mathcal{L}}_\alpha^y}{V_y^y}) \\ &= \frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x |t|}{1-V_x^x |t|}} e^{\frac{\hat{\mathcal{L}}_\alpha^y |s|}{1-V_y^y |s|}}}{(1 - |t| V_x^x)^\alpha (1 - |s| V_y^y)^\alpha} \end{aligned}$$

By Proposition 4.5, for $|s|$ and $|t|$ small enough, $\mathbb{E} \left[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x |t|}{1-V_x^x |t|}} e^{\frac{\hat{\mathcal{L}}_\alpha^y |s|}{1-V_y^y |s|}}}{(1 - |t| V_x^x)^\alpha (1 - |s| V_y^y)^\alpha} \right] < \infty$. Consequently,

$$\begin{aligned} \sum_{k,l \in \mathbb{N}} t^k P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x) s^l P_l^{\alpha, V_y^y}(\hat{\mathcal{L}}_\alpha^y) &= \mathbb{E} \left[\frac{e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+t V_x^x}} e^{\frac{s \hat{\mathcal{L}}_\alpha^y}{1+s V_y^y}}}{(1 + t V_x^x)^\alpha (1 + s V_y^y)^\alpha} \right] \\ &= (1 - st V_y^x V_x^y)^{-\alpha} = \sum_{k \in \mathbb{N}} \binom{\alpha + k}{k} (st V_y^x V_x^y)^k. \end{aligned}$$

Finally, identifying the coefficients of $s^k t^l$, we obtain

$$\mathbb{E}[P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x) P_l^{\alpha, V_y^y}(\hat{\mathcal{L}}_\alpha^y)] = \delta_l^k \binom{\alpha + k}{k} (V_y^x V_x^y)^k.$$

□

Proposition 4.10. Fix some $p \geq 1$, for $|t|$ small enough, $\alpha \rightarrow (1 + t V_x^x)^{-\alpha} e^{\frac{\hat{\mathcal{L}}_\alpha^x t}{1+t V_x^x}}$ and $\alpha \rightarrow P_k^{\alpha, V_x^x}(\hat{\mathcal{L}}_\alpha^x)$ are continuous L^p -martingales indexed by $\alpha > 0$.

4.3 Limit behavior of the occupation field

Remark 8. $(X_\alpha = (\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}), \alpha \geq 0)$ is a multi-subordinator with respect to the increasing family of σ -fields $\mathcal{F}_\alpha = \sigma(\mathcal{L}_s, s \leq \alpha)$.

$\mathbb{E}[\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n}] = (V_{x_1}^{x_1}, \dots, V_{x_n}^{x_n})$ and $\mathbb{E}[e^{-\lambda_1 \hat{\mathcal{L}}_\alpha^{x_1} - \dots - \lambda_n \hat{\mathcal{L}}_\alpha^{x_n}}] = e^{-\alpha \Phi(\lambda_1, \dots, \lambda_n)}$, where

$$\Phi(\lambda_1, \dots, \lambda_n) = \int_{y^1, \dots, y^n \in \mathbb{R}^+} (1 - e^{-\sum_{i=1}^n \lambda_i y^i}) \mu(l^{x_1} \in dy^1, \dots, l^{x_n} \in dy^n).$$

So $\lim_{\alpha \rightarrow \infty} \frac{1}{\alpha} (\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}) = (V_{x_1}^{x_1}, \dots, V_{x_n}^{x_n})$. And $(\frac{\hat{\mathcal{L}}_\alpha^{x_1} - \alpha V_{x_1}^{x_1}}{\sqrt{\alpha}}, \dots, \frac{\hat{\mathcal{L}}_\alpha^{x_n} - \alpha V_{x_n}^{x_n}}{\sqrt{\alpha}})$ converges in law to a Gaussian variable with mean 0 and covariance $(C_{ij} = V_{x_j}^{x_i} V_{x_i}^{x_j}, i, j = 1, \dots, n)$.

The following result comes from [dA94]: the rescaled Lévy process $(\frac{1}{t} X(ts), s \geq 0), t > 0$ verifies the strong large deviation principle with a good rate function as $t \rightarrow \infty$ under the exponential integrability condition:

$$\exists \beta > 0, \mathbb{E}[e^{\beta \|X(1)\|}] < \infty$$

This is true for the subordinator $((\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}), \alpha > 0)$ by Proposition 4.5. The proposition below follows by application of the contraction principle.

Proposition 4.11. $\frac{1}{\alpha}(\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}) \in \mathbb{R}^n$ verifies a strong large derivation principle with good rate function $\Lambda^* : \mathbb{R}^n \rightarrow [0, \infty]$ when α tends to ∞ . Here, $\Lambda(u) = \ln \mathbb{E}[e^{\hat{\mathcal{L}}_1^{x_1} u_1 + \dots + \hat{\mathcal{L}}_1^{x_n} u_n}]$ and $\Lambda^*(y) = \sup_{u \in \mathbb{R}^d} (\langle u, y \rangle - \Lambda(u))$.

To be more precise, for all open set $O \subset \mathbb{R}^n$,

$$\liminf_{\alpha \rightarrow \infty} \frac{1}{\alpha} \ln(\mathbb{P}[\frac{1}{\alpha}(\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}) \in O]) \geq - \inf_{y \in O} \Lambda^*(y)$$

and for all closed subset C of \mathbb{R}^n ,

$$\limsup_{\alpha \rightarrow \infty} \frac{1}{\alpha} \ln(\mathbb{P}[\frac{1}{\alpha}(\hat{\mathcal{L}}_\alpha^{x_1}, \dots, \hat{\mathcal{L}}_\alpha^{x_n}) \in C]) \leq - \inf_{y \in C} \Lambda^*(y).$$

Remark 9. In particular, for $n = 1$,

$$\Lambda^*(y) = \begin{cases} \ln(\frac{V_x^x}{y}) - 1 + \frac{y}{V_x^x} & y > 0 \\ \infty & y \leq 0 \end{cases}$$

For $n = 2$,

$$\Lambda^*(y) = \begin{cases} \ln\left(\frac{1 + \sqrt{1 + \frac{4y^1 y^2 V_{x_1}^{x_1} V_{x_2}^{x_2}}{\det(V|_{\{x_1, x_2\}})^2}}}{2y^1 y^2}\right) + \ln \det(V|_{\{x_1, x_2\}}) \\ + \frac{y^1 V_{x_2}^{x_2} + y^2 V_{x_1}^{x_1}}{\det(V|_{\{x_1, x_2\}})} - 1 - \sqrt{1 + \frac{4y^1 y^2 V_{x_1}^{x_1} V_{x_2}^{x_2}}{\det(V|_{\{x_1, x_2\}})^2}} & y_1, y_2 \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof of the remark. For $n = 1$, by Proposition 4.5, $\Lambda(u) = -\ln(1 - uV_x^x)$ for $u < \frac{1}{V_x^x}$ and $\Lambda(u) = \infty$ otherwise. Then,

$$\Lambda^*(y) = \sup_{u \in \mathbb{R}} (uy - \Lambda(u)) = \begin{cases} \frac{y - V_x^x}{yV_x^x} y - \Lambda(\frac{y - V_x^x}{yV_x^x}) & y > 0 \\ \infty & y \leq 0 \end{cases} = \begin{cases} \ln(\frac{V_x^x}{y}) - 1 + \frac{y}{V_x^x} & y > 0 \\ \infty & y \leq 0. \end{cases}$$

Denote by $A(u_1, u_2)$ the matrix $\begin{bmatrix} V_{x_1}^{x_1} u_1 & V_{x_2}^{x_1} \sqrt{u_1 u_2} \\ V_{x_1}^{x_2} \sqrt{u_1 u_2} & V_{x_2}^{x_2} u_2 \end{bmatrix}$.

For $n = 2$, by Proposition 4.5, for $u_1 \geq 0, u_2 \geq 0$,

$$\mathbb{E}[e^{\hat{\mathcal{L}}_1^{x_1} u_1 + \hat{\mathcal{L}}_1^{x_2} u_2}] = \begin{cases} 1/\det(I - A(u_1, u_2)) & \text{if } 1 < 1/\rho(A(u_1, u_2)) \\ \infty & \text{otherwise} \end{cases}$$

where the spectral radius

$$\rho(A(u_1, u_2)) = \frac{V_{x_1}^{x_1} u_1 + V_{x_2}^{x_2} u_2 + \sqrt{(V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2)^2 + 4V_{x_2}^{x_1} V_{x_1}^{x_2} u_1 u_2}}{2}$$

and

$$1/\det(I - A(u_1, u_2)) = \frac{1}{1 - V_{x_1}^{x_1} u_1 - V_{x_2}^{x_2} u_2 + \det(V|_{\{x_1, x_2\}}) u_1 u_2}.$$

Finally, for $u_1, u_2 \geq 0$,

if $1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2 > 0$ and $V_{x_1}^{x_1}u_1 + V_{x_2}^{x_2}u_2 < 2$,

$$\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise.

For $u_1, u_2 \leq 0$, $\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2)$.

For $u_1 > 0, u_2 < 0$,

$$\mathbb{E}[e^{\hat{\mathcal{L}}_1^{x_1}u_1 + \hat{\mathcal{L}}_1^{x_1}u_2}] = \exp\left(\int (e^{u_1 l^{x_1}} - 1)e^{u_2 l^{x_2}}\mu(dl) + \int (e^{u_2 l^{x_2}} - 1)\mu(dl)\right).$$

By Proposition 3.27, $\int (e^{u_2 l^{x_2}} - 1)\mu(dl) = -\ln \det(1 - u^2 V_{x_2}^{x_2})$. By Theorem 3.19 and then Proposition 3.27,

$$\begin{aligned} \int (e^{u_1 l^{x_1}} - 1)e^{u_2 l^{x_2}}\mu(dl) &= \int (e^{u_1 l^{x_1}} - 1)\mu(L - M_{-u_2 \delta_{x_2}}, dl) \\ &= \begin{cases} -\ln(1 - u_1(V_{-u_2 \delta_{x_2}})_{x_1}^{x_1}) & \text{if } u_1 < 1/(V_{-u_2 \delta_{x_2}})_{x_1}^{x_1} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

By the resolvent equation,

$$V_{x_2}^{x_1} = (V_{-u_2 \delta_{x_2}})_{x_2}^{x_1} + (V_{-u_2 \delta_{x_2}})_{x_2}^{x_1}(-u_2)V_{x_2}^{x_2}.$$

Therefore, $(V_{-u_2 \delta_{x_2}})_{x_2}^{x_1} = \frac{V_{x_2}^{x_1}}{1 - u_2 V_{x_2}^{x_2}}$. Again, by the resolvent equation,

$$V_{x_1}^{x_1} = (V_{-u_2 \delta_{x_2}})_{x_1}^{x_1} + (V_{-u_2 \delta_{x_2}})_{x_2}^{x_1}(-u_2)V_{x_1}^{x_2}.$$

We deduce that $(V_{-u_2 \delta_{x_2}})_{x_1}^{x_1} = V_{x_1}^{x_1} + \frac{u_2 V_{x_2}^{x_1} V_{x_1}^{x_2}}{1 - u_2 V_{x_2}^{x_2}}$. Therefore, for $u_1 > 0, u_2 < 0$,

if $1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2 > 0$,

$$\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise. It is easy to check that $V_{x_1}^{x_1}u_1 + V_{x_2}^{x_2}u_2 < 2$ is implied by

$1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2 > 0$ for $u_1 > 0, u_2 < 0$. Similar results can be proved

for $u_1 < 0, u_2 > 0$. In the end, for any $u_1, u_2 \in \mathbb{R}$,

if $1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2 > 0$ and $V_{x_1}^{x_1}u_1 + V_{x_2}^{x_2}u_2 < 2$,

$$\Lambda(u_1, u_2) = -\ln(1 - V_{x_1}^{x_1}u_1 - V_{x_2}^{x_2}u_2 + \det(V|_{\{x_1, x_2\}})u_1u_2)$$

and $\Lambda(u_1, u_2) = \infty$ otherwise.

It is obvious that $\Lambda^*(y_1, y_2) = \infty$ for $y_1 \leq 0$ or $y_2 \leq 0$. Fixing $y_1, y_2 > 0$, we are

able to solve $(\frac{\partial}{\partial u_1}\Lambda(u_1, u_2), \frac{\partial}{\partial u_2}\Lambda(u_1, u_2)) = (y_1, y_2)$. We find that the extreme value of

$\langle u, y \rangle - \Lambda(u)$ is reached for

$$u_1 = \frac{V_{x_2}^{x_2}}{V_{x_1}^{x_1}V_{x_2}^{x_2} - V_{x_2}^{x_1}V_{x_1}^{x_2}} - \frac{1 + \sqrt{1 + \frac{4y_1y_2V_{x_2}^{x_1}V_{x_1}^{x_2}}{(V_{x_1}^{x_1}V_{x_2}^{x_2} - V_{x_2}^{x_1}V_{x_1}^{x_2})^2}}}{2y_1}$$

$$u_2 = \frac{V_{x_1}^{x_1}}{V_{x_1}^{x_1} V_{x_2}^{x_2} - V_{x_2}^{x_1} V_{x_1}^{x_2}} - \frac{1 + \sqrt{1 + \frac{4y_1 y_2 V_{x_2}^{x_1} V_{x_1}^{x_2}}{(V_{x_1}^{x_1} V_{x_2}^{x_2} - V_{x_2}^{x_1} V_{x_1}^{x_2})^2}}}{2y_2}$$

and then conclude that

$$\begin{aligned} \Lambda^*(y) = \ln \left(\frac{1 + \sqrt{1 + \frac{4y^1 y^2 V_{x_1}^{x_1} V_{x_2}^{x_2}}{\det(V|_{\{x_1, x_2\}})^2}}}{2y^1 y^2} \right) + \ln \det(V|_{\{x_1, x_2\}}) \\ + \frac{y^1 V_{x_2}^{x_2} + y^2 V_{x_1}^{x_1}}{\det(V|_{\{x_1, x_2\}})} - 1 - \sqrt{1 + \frac{4y^1 y^2 V_{x_1}^{x_1} V_{x_2}^{x_2}}{\det(V|_{\{x_1, x_2\}})^2}}. \end{aligned}$$

□

4.4 Hitting probabilities

Definition 4.7. For $D \subset S$, define $loop^D = \{l; \langle l, 1_{\{S-D\}} \rangle = 0\}$, namely loops contained in D . Let $\mathcal{L}_\alpha^D = \mathcal{L}_\alpha \cap loop^D$ be the restriction of the Poisson ensemble on $loop^D$.

Since $\mu(\{l; l \text{ is a trivial loop at } x\}) = \infty$, \mathcal{L}_α contains infinitely many trivial loops at x $\mu - a.s.$

Proposition 4.12. For a finite subset F ,

$$\mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ is non-trivial and } l \text{ visits } F] = \left(\prod_{x \in F} (-L_x^x) \det(V_F) \right)^{-\alpha}.$$

Proof. It is a direct consequence of Proposition 3.33 and the definition of the Poisson random measure. □

Remark 10. For any subset F , we can find $F_1 \subset \dots \subset F_n \subset \dots$ a sequence of finite subsets of F increasing to F . Then,

$$\begin{aligned} \mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ is non-trivial and } l \text{ visits } F] \\ = \lim_{n \rightarrow \infty} \downarrow \mathbb{P}[\nexists l \in \mathcal{L}_\alpha; l \text{ is non-trivial and } l \text{ visits } F_n] \\ = \lim_{n \rightarrow \infty} \downarrow \left(\prod_{x \in F_n} (-L_x^x) \det(V_{F_n}) \right)^{-\alpha} \\ = \inf_{A \subset F, |A| < \infty} \left(\prod_{x \in A} (-L_x^x) \det(V_A) \right)^{-\alpha}. \end{aligned}$$

4.5 Densities of the occupation field

A non-symmetric generalization of Dynkin's isomorphism was given in [LJ08]. Suppose L is the generator of a transient sub-Markovian process on $\{x_1, \dots, x_n\}$, m is an excessive

measure, and χ is a non-negative function on $\{x_1, \dots, x_n\}$, then

$$\frac{1}{(2\pi)^n} \int e^{-\frac{1}{2}\langle z\bar{z}, \chi \rangle_m} e^{\frac{1}{2}\langle Lz, z \rangle_m} \prod du^i dv^i = \det(-M_m L + M_{\chi m})^{-1}$$

where $z^j = u^j + \sqrt{-1} \cdot v^j$ for $j = 1, \dots, n$ and $L_j^i = L_{x_j}^{x_i}$ for $i, j = 1, \dots, n$. And it has been proved that if $\text{supp}(\chi) \subset F$, then $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_1, \chi \rangle}] = \frac{\det(V_F)_\chi}{\det(V_F)} = \frac{\det(-L_F)}{\det(-L_F + \chi)}$. So, we have the following representation.

Proposition 4.13. *Let $F = \{x_1, \dots, x_n\} \subset S$ and $L_F = (-V_F)^{-1}$. For any bounded measurable function G ,*

$$\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})] = \frac{\det(-M_m L_F)}{(2\pi)^n} \int G\left(\frac{1}{2}m_1|z^1|^2, \dots, \frac{1}{2}m_n|z^n|^2\right) e^{\frac{1}{2}\langle L_F z, z \rangle_m} \prod du^i dv^i.$$

Remark 11. Recall that in the symmetric case, if ϕ is a Gaussian free field with covariance matrix given by the Green function, $\hat{\mathcal{L}}_{1/2}$ has the same law as $\frac{1}{2}\phi^2$. Moreover, if ϕ_1, \dots, ϕ_k are k i.i.d. copies of ϕ , then $\frac{1}{2} \sum_{j=1}^k \phi_k^2$ and $\hat{\mathcal{L}}_{k/2}$ have the same law. For details, see Chapter 5 in [LJ11] and Chapter 4 in [Szn12].

We can derive from this expression a formula for the joint densities of the occupation field, for $\alpha = 1$.

Proposition 4.14. *Let $F = \{x_1, \dots, x_n\} \subset S$ and $L_F = (-V_F)^{-1}$. Then, $f(\rho^1, \dots, \rho^n)$, the density of $(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})$ with respect to the Lebesgue measure on \mathbb{R}_+^n is*

$$\det(-L_F) \sum_{n_{ij} \in \mathbb{N}, i, j=1, \dots, n} 1_{\{\sum_j n_{ij} = \sum_j n_{ji}, i=1, \dots, n\}} \left(\prod_{i, j=1, \dots, n} \frac{((L_F)_{x_j}^{x_i} \sqrt{\rho^i \rho^j})^{n_{ij}}}{n_{ij}!} \right).$$

Proof. The above Proposition shows that for any G bounded measurable

$$\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})] = \frac{\det(-M_m L_F)}{(2\pi)^n} \int G\left(\frac{1}{2}m_1|z^1|^2, \dots, \frac{1}{2}m_n|z^n|^2\right) e^{\frac{1}{2}\langle L_F z, z \rangle_m} \prod du^i dv^i.$$

Using the polar coordinate, let $r^j = |z^j|$, $\theta_j \in [0, 2\pi[$, $u^j = r^j \cos(\theta_j)$ and $v^j = r^j \sin(\theta_j)$.

Then, $\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})]$ equals

$$\frac{\det(-M_m L_F)}{(2\pi)^n} \int G\left(\frac{1}{2}m_1(r^1)^2, \dots, \frac{1}{2}m_n(r^n)^2\right) e^{\sum_{i,j} \frac{1}{2}(L_F)_{x_j}^{x_i} r^i r^j m_j e^{i\theta_i} e^{-i\theta_j}} \prod dr^i d\theta_i.$$

Let $\rho^j = m_j(r^j)^2/2$ for $j = 1, \dots, n$, then $\mathbb{E}[G(\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n})]$ equals

$$\frac{\det(-L_F)}{(2\pi)^n} \int G(\rho^1, \dots, \rho^n) e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i \rho^j \frac{m_i}{m_j}\right)^{1/2} e^{i\theta_i} e^{-i\theta_j}} \prod d\rho^i d\theta_i.$$

Therefore, the density of $\hat{\mathcal{L}}_1^{x_1}, \dots, \hat{\mathcal{L}}_1^{x_n}$, is

$$f(\rho^1, \dots, \rho^n) = \int_{[0, 2\pi]^n} \frac{\det(-L_F)}{(2\pi)^n} e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i \rho^j \frac{m_i}{m_j}\right)^{1/2} e^{i\theta_i} e^{-i\theta_j}} \prod d\theta_i$$

$$= \int_{|z^i|=1, i=1, \dots, n} \frac{\det(-L_F)}{(2\pi i)^n} \frac{1}{z^1 \dots z^n} e^{\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i \rho^j \frac{m_i}{m_j} \right)^{1/2} \frac{z^i}{z^j}} dz^1 \dots dz^n.$$

We expand $\exp \left(\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i \rho^j \frac{m_i}{m_j} \right)^{1/2} \frac{z^i}{z^j} \right)$ into series, integrate it term by term and use Cauchy's formula. Only the constant terms in the expansion of

$$\exp \left(\sum_{i,j} (L_F)_{x_j}^{x_i} \left(\rho^i \rho^j \frac{m_i}{m_j} \right)^{1/2} \frac{z^i}{z^j} \right)$$

contribute. Accordingly, we have

$$f(\rho^1, \dots, \rho^n) = \det(-L_F) \sum_{\substack{n_{ij} \in \mathbb{N} \text{ for } i,j=1, \dots, n \\ \sum_j n_{ij} = \sum_j n_{ji} \text{ for } i=1, \dots, n}} \left(\prod_{i,j=1, \dots, n} \frac{((L_F)_{x_j}^{x_i} \sqrt{\rho^i \rho^j})^{n_{ij}}}{n_{ij}!} \right).$$

□

Moreover, we have the follow expansions of the density of occupation field for general $\alpha > 0$:

Proposition 4.15. *Denote by $\text{Coeff}(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \dots s_n^{M_n})^3$ the coefficient before the term $s_1^{M_1} \dots s_n^{M_n}$ in the expansion of the function $s \rightarrow \det(M_s + V_F)^{-\alpha}$ for s small enough. Then the density $(f^\alpha(\rho_1, \dots, \rho_n), \rho_1, \dots, \rho_n > 0)$ of the occupation field $(\mathcal{L}_\alpha^{x_1}, \dots, \mathcal{L}_\alpha^{x_n})$ has the following expression:*

$$f^\alpha(\rho_1, \dots, \rho_n) = \sum_{M_1, \dots, M_n \in \mathbb{N}} \frac{\text{Coeff}(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \dots s_n^{M_n})}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \prod_{i=1}^n \rho_i^{M_i + \alpha - 1}.$$

Proof. Let's calculate the Laplace transform of the function

$$(\rho_1, \dots, \rho_n) \rightarrow f^\alpha(\rho_1, \dots, \rho_n) e^{-c(\rho_1 + \dots + \rho_n)}.$$

For c sufficient large, we have

$$\sum_{M_1, \dots, M_n \in \mathbb{N}} \frac{|\text{Coeff}(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \dots s_n^{M_n})|}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \prod_{i=1}^n \rho_i^{M_i + \alpha - 1} e^{-c(\rho_1 + \dots + \rho_n)} < \infty$$

$$\begin{aligned} & \int d\rho_1 \dots d\rho_n e^{-(\rho_1 \lambda_1 + \dots + \rho_n \lambda_n)} f^\alpha(\rho_1, \dots, \rho_n) e^{-c(\rho_1 + \dots + \rho_n)} \\ &= \sum_{M_1, \dots, M_n \in \mathbb{N}} \frac{\text{Coeff}(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \dots s_n^{M_n})}{\prod_{i=1}^n \Gamma(M_i + \alpha)} \int \prod_{i=1}^n \rho_i^{M_i + \alpha - 1} e^{-\rho_i(\lambda_i + c)} d\rho_i \end{aligned}$$

³For s sufficient close to $(0, \dots, 0)$, $\det(M_s + V_F)^{-\alpha}$ is an analytic function.

$$\begin{aligned}
&= \sum_{M_1, \dots, M_n \in \mathbb{N}} \frac{\text{Coeff}(\det(M_s + V_F)^{-\alpha}, s_1^{M_1} \dots s_n^{M_n})}{\prod_{i=1}^n (\lambda_i + c)^{M_i + \alpha}} \\
&= \det(M_{\lambda+c}^{-1} + V_F)^{-\alpha} \prod_{i=1}^n (\lambda_i + c)^{-\alpha} \\
&= \det(I + M_{\lambda+c} V_F)^{-\alpha} = \mathbb{E}[e^{-\sum_{i=1}^n \mathcal{L}_\alpha^{x_i}(\lambda_i + c)}].
\end{aligned}$$

Clearly, f^α is the density of $(\mathcal{L}_\alpha^{x_1}, \dots, \mathcal{L}_\alpha^{x_n})$. \square

4.6 Conditioned occupation field

Definition 4.8. For $F \subset S$, define $\mathcal{L}_\alpha|_F = \{l_F : l \in \mathcal{L}_\alpha\}$ where l_F is the trace of l on F , see Definition 3.18.

Proposition 4.16. Let X, Y be two Borel spaces. Let \mathcal{P} be a Poisson random measure on $Z = X \times Y$ with σ -finite intensity measure $\mu(dx, dy) = m(dx)K(x, dy)$, K being a probability kernel. Let π_X and π_Y be the projection from $Z = X \times Y$ to X and Y respectively. Define $\mathcal{P}_X = \pi_X \circ \mathcal{P}$ and $\mathcal{P}_Y = \pi_Y \circ \mathcal{P}$. For all $\Phi : Y \rightarrow \mathbb{R}$ non-negative measurable, define $\phi : Y \rightarrow \mathbb{R}$ according to Φ by the following equation $e^{-\phi(x)} = \int_Y e^{-\Phi(y)} K(x, dy)$. Then,

$$\mathbb{E}[e^{-\langle \mathcal{P}_Y, \Phi \rangle} | \mathcal{F}_X] = e^{-\langle \mathcal{P}_X, \phi \rangle}.$$

Remark 12. The Poisson random measure \mathcal{P} is the K -randomization⁴ of the Poisson random measure $\pi_X \circ \mathcal{P}$.

Proof. Take $\Psi : X \rightarrow \mathbb{R}$ and $\Phi : Y \rightarrow \mathbb{R}$ non-negative measurable. Define ϕ by the following equation:

$$e^{-\phi(x)} = \int_Y e^{-\Phi(y)} K(x, dy).$$

We have

$$\begin{aligned}
\mathbb{E}[e^{-\langle \mathcal{P}_X, \Psi \rangle} e^{-\langle \mathcal{P}_Y, \Phi \rangle}] &= \mathbb{E}[e^{-\langle \mathcal{P}, \Psi \otimes \Phi \rangle}] = e^{\mu(e^{-\Psi \otimes \Phi} - 1)} \\
&= \exp\left(\int_{X \times Y} (e^{-\Psi(x)} e^{-\Phi(y)} - 1) m(dx) K(x, dy)\right) \\
&= \exp\left(\int_X (e^{-\Psi(x)} \int_Y e^{-\Phi(y)} K(x, dy) - 1) m(dx)\right) \\
&= \exp\left(\int_X (e^{-\Psi(x) - \phi(x)} - 1) m(dx)\right) = \mathbb{E}[e^{-\langle \mathcal{P}_X, \Psi \rangle} e^{-\langle \mathcal{P}_Y, \phi \rangle}].
\end{aligned}$$

⁴Please refer to Chapter 12 of [Kal02]

Since $\mathcal{F}_X = \sigma(\{e^{-\langle \mathcal{P}_X, \Psi \rangle} : \Psi \text{ is a non-negative measurable function on } X\})$,

$$\mathbb{E}[e^{-\langle \mathcal{P}_Y, \Phi \rangle} | \mathcal{F}_X] = e^{-\langle \mathcal{P}_X, \Phi \rangle}.$$

□

Let f be a positive measurable function on the space of excursions. Recall that $\mathcal{E}_F(l)$ is the point measure of the excursions of the loop l outside of F (see Definition 3.17). As a consequence of Proposition 4.16 and Proposition 3.15 or Corollary 3.16, we have the following proposition.

Proposition 4.17.

$$\mathbb{E}[e^{-\sum_{l \in \mathcal{L}_\alpha} \langle \mathcal{E}_F(l), f \rangle} | \sigma(\mathcal{L}_\alpha | F)] = \left(\prod_{x \neq y \in F} \nu_{F,ex}^{x,y} (e^{-f})^{N_y^x(\mathcal{L}_\alpha | F)} \right) \times e^{\sum_{x \in F} (L_x^x - (L_F)_x^x) (\widehat{\mathcal{L}_\alpha | F})^x \nu_{F,ex}^{x,x} (1 - e^{-f})}.$$

For an excursion (e, x, y) outside of F from x to y and χ any non-negative measurable function on S , set $\langle e, \chi \rangle = \int \chi(e(s)) ds$. Then we have the following:

Proposition 4.18. *The conditional expectation $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle} | \sigma(\mathcal{L}_\alpha | F)]$ equals*

$$\mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] e^{-\langle \hat{\mathcal{L}}_\alpha | F, \chi \rangle} \exp \left(\sum_{x \in F} L_x^x (\widehat{\mathcal{L}_\alpha | F})^x ((R^F)_x^x - (R_\chi^F)_x^x) \right) \prod_{x \neq y \in F} \left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x} \right)^{N_y^x(\mathcal{L}_\alpha | F)}.$$

Proof. The set of loops which do not intersect F , $\mathcal{L}_\alpha^{F^c}$, is independent of the set of loops which intersect F . Therefore,

$$\begin{aligned} \mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle} | \sigma(\mathcal{L}_\alpha | F)] &= \mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] \mathbb{E}[\exp(-\sum_{\substack{l \in \mathcal{L}_\alpha \\ l \text{ visits } F}} \langle l, \chi \rangle) | \sigma(\mathcal{L}_\alpha | F)] \\ &= \mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] \mathbb{E}[\exp(-\sum_{\substack{l \in \mathcal{L}_\alpha \\ l \text{ visits } F}} (\langle l_F, \chi \rangle + \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle)) | \sigma(\mathcal{L}_\alpha | F)] \\ &= \mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] \mathbb{E}[\exp(-\sum_{\substack{l \in \mathcal{L}_\alpha \\ l \text{ visits } F}} \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle) \exp(-\langle \hat{\mathcal{L}}_\alpha | F, \chi \rangle) | \sigma(\mathcal{L}_\alpha | F)] \\ &= \mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] e^{-\langle \hat{\mathcal{L}}_\alpha | F, \chi \rangle} \mathbb{E}[\exp(-\sum_{\substack{l \in \mathcal{L}_\alpha \\ l \text{ visits } F}} \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle) | \sigma(\mathcal{L}_\alpha | F)] \\ &= \mathbb{E}[e^{-\langle \widehat{\mathcal{L}}_\alpha^{F^c}, \chi \rangle}] e^{-\langle \hat{\mathcal{L}}_\alpha | F, \chi \rangle} \mathbb{E}[\exp(-\sum_{l \in \mathcal{L}_\alpha} \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle) | \sigma(\mathcal{L}_\alpha | F)]. \end{aligned}$$

By Proposition 4.17, taking the positive excursion function $f(\cdot)$ to be $\langle \cdot, \chi \rangle$,

$$\begin{aligned} \mathbb{E}[e^{-\sum_{l \in \mathcal{L}_\alpha} \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle} | \sigma(\mathcal{L}_\alpha | F)] &= \mathbb{E}[e^{-\sum_{l \in \mathcal{L}_\alpha} \langle \mathcal{E}_F(l), \langle \cdot, \chi \rangle \rangle} | \sigma(\mathcal{L}_\alpha | F)] \\ &= \left(\prod_{x \neq y \in F} \nu_{F,ex}^{x,y} (e^{-\langle \cdot, \chi \rangle})^{N_y^x(\mathcal{L}_\alpha | F)} \right) \exp \left(\sum_{x \in F} (L_x^x - (L_F)_x^x) (\widehat{\mathcal{L}_\alpha | F})^x \nu_{F,ex}^{x,x} (1 - e^{-\langle \cdot, \chi \rangle}) \right). \end{aligned}$$

By Lemma 3.17,

$$\nu_{F,ex}^{x,y}(e^{-\langle \cdot, \chi \rangle}) = \frac{(R_\chi^F)_y^x}{(R^F)_y^x}.$$

Then, by Proposition 2.6, $L_x^x - (L_F)_x^x = L_x^x (R^F)_x^x$. Then,

$$\begin{aligned} \mathbb{E}[e^{-\sum_{l \in \mathcal{L}_\alpha} \sum_{e \in \mathcal{E}_F(l)} \langle e, \chi \rangle} | \sigma(\mathcal{L}_\alpha|_F)] \\ = \left(\prod_{x \neq y \in F} \left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x} \right)^{N_y^x(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} (L_x^x - (L_F)_x^x) \widehat{(\mathcal{L}_\alpha|_F)}^x \left(1 - \frac{(R_\chi^F)_x^x}{(R^F)_x^x} \right) \right) \\ = \left(\prod_{x \neq y \in F} \left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x} \right)^{N_y^x(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} L_x^x \widehat{(\mathcal{L}_\alpha|_F)}^x ((R^F)_x^x - (R_\chi^F)_x^x) \right). \end{aligned}$$

Finally, we get $\mathbb{E}[e^{-\langle \hat{\mathcal{L}}_\alpha, \chi \rangle} | \sigma(\mathcal{L}_\alpha|_F)]$ equals

$$\mathbb{E}[e^{-\langle \widehat{\mathcal{L}_\alpha^F}, \chi \rangle}] e^{-\langle \hat{\mathcal{L}}_\alpha|_F, \chi \rangle} \left(\prod_{x \neq y \in F} \left(\frac{(R_\chi^F)_y^x}{(R^F)_y^x} \right)^{N_y^x(\mathcal{L}_\alpha|_F)} \right) \exp \left(\sum_{x \in F} L_x^x \widehat{(\mathcal{L}_\alpha|_F)}^x ((R^F)_x^x - (R_\chi^F)_x^x) \right).$$

□

4.7 Loop clusters

Consider the space S as a graph (S, E) with S as the set of vertices and $E = \{\{x, y\} : N_y^x(l) > 0 \text{ or } N_x^y(l) > 0\}$ as the set of undirected edges. An edge $\{x, y\}$ is said to be open at time α if it is traversed by at least one loop of \mathcal{L}_α , i.e. $N_y^x(\mathcal{L}_\alpha) + N_x^y(\mathcal{L}_\alpha) > 0$. The set of open edges defines a subgraph G_α with vertices S . The connected components of G_α define a partition of S denoted by \mathcal{C}_α , namely the loop clusters at time α .

As in section 2 of [LJL12], we have the following proposition,

Proposition 4.19. *Given a collection of edges $F = \{e_1 = \{x_1, y_1\}, \dots, e_k = \{x_k, y_k\}\}$, let $A = \bigcup_{i=1}^k \{x_i, y_i\}$. Then,*

$$\mathbb{P}[e_1, \dots, e_k \text{ are all closed}] = \det(I + (L|_F)|_{A \times A} V_A)^{-\alpha}$$

$$\text{where } (L|_F)_y^x = \begin{cases} L_y^x & \text{if } \{x, y\} \in F \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose S is finite,

$$\begin{aligned} \mathbb{P}[e_1, \dots, e_k \text{ are all closed}] &= \exp(-\alpha \mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0)) \\ &= \exp(-\alpha \mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0, l \text{ is non-trivial})) \end{aligned}$$

$$= \exp(-\alpha\mu(l \text{ is non-trivial}) + \alpha\mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) = 0, l \text{ is non-trivial}))$$

Define $(L')_y^x = L_y^x$ if $\{x, y\} \notin F$ and $(L')_y^x = 0$ if $\{x, y\} \in F$. By Proposition 3.3,

$$\mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) = 0, l \text{ is non-trivial}) = \mu(L', l \text{ is non-trivial}).$$

(Recall that $\mu(L', dl)$ is the Markovian loop measure associated with the generator L' .) By Proposition 3.33, $\mu(L', l \text{ is non-trivial}) = -\ln(\prod_{x \in S} (-L')_x^x) + \ln \det(-L') = \ln(\prod_{x \in S} (-L)_x^x) - \ln \det(-L')$ and $\mu(l \text{ is non-trivial}) = \ln(\prod_{x \in S} (-L)_x^x) - \ln \det(-L)$. Therefore,

$$\mathbb{P}[e_1, \dots, e_k \text{ are all closed}] = \left(\frac{\det(-L)}{\det(-L')} \right)^\alpha = \det(-L'V)^{-\alpha}.$$

Write as $-L' = -L + (L - L') = -L + L|_F$. Therefore, $\det(-L'V) = \det(I + (L|_F)|_{A \times A} V_A)$. Consequently,

$$\mathbb{P}[e_1, \dots, e_k \text{ are all closed}] = \det(I + (L|_F)|_{A \times A} V_A)^{-\alpha}.$$

For S countable, let $A_1 \subset A_2 \subset \dots$ exhausting S . Then we have

$$\begin{aligned} \mathbb{P}[e_1, \dots, e_k \text{ are all closed}] &= \exp(-\alpha\mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0)) \\ &= \exp(-\alpha \lim_{n \rightarrow \infty} \mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0, l \text{ is contained in } A_n)). \end{aligned}$$

By Proposition 3.9,

$$\mu(\sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0, l \text{ is contained in } A_n) = \mu(L|_{A_n \times A_n}, \sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0).$$

By the calculation for the finite case,

$$\mu(L|_{A_n \times A_n}, \sum_{i=1}^k N_{y_i}^{x_i}(l) + N_{x_i}^{y_i}(l) > 0) = \det(I + (L|_F)|_{A \times A} (-L|_{A_n \times A_n})_A^{-1})^{-\alpha}.$$

It is not hard to check that $\lim_{n \rightarrow \infty} ((-L|_{A_n \times A_n})^{-1})_y^x = V_y^x$ for $x, y \in S$. Finally,

$$\mathbb{P}[e_1, \dots, e_k \text{ are all closed}] = \det(I + (L|_F)|_{A \times A} V_A)^{-\alpha}.$$

□

As a corollary, we obtain another expression by using the Poisson kernel. For $X \subset S$, define the Poisson kernel $(H^X)_y^x = \mathbb{P}^x[X_{T_X} = y]$ the probability of hitting X at the position y for a process starting from x .

Proposition 4.20. *Given a partition $\pi = \{S_1, \dots, S_k\}$, define $\partial S_i = \{x \in S_i : \exists y \in S_i^c, Q_y^x + Q_x^y > 0\}$, $F = \bigcup_{i=1}^k \{\{x, y\} : Q_y^x + Q_x^y > 0, x \in S_i, y \in S_j\}$ and $A = \bigcup_{i=1}^k \partial S_i$. Suppose $|A| < \infty$. Define $H_{i,j} = H_{S_i, S_j}^{S_i^c}|_{\partial S_i \times \partial S_j}$ and*

$$K = \begin{bmatrix} 0 & H_{1,2} & \cdots & H_{1,k} \\ H_{2,1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & H_{k-1,k} \\ H_{k,1} & \cdots & H_{k,k-1} & 0 \end{bmatrix}.$$

Then,

$$\mathbb{P}[\mathcal{C}_\alpha \text{ is finer than } \pi] = \mathbb{P}[\text{all the edges in } F \text{ are closed}] = (\det(I - K))^\alpha.$$

Proof. By taking the trace of the loops on A , we can suppose the state space S is finite and $\partial S_i = S_i$ for $i = 1, \dots, k$. By an argument similar to the argument in the above proposition, we see that

$$\mathbb{P}[\mathcal{C}_\alpha \text{ is finer than } \pi] = \left(\frac{\det(L')}{\det(L)} \right)^{-\alpha}$$

where $(L')_y^x = L_y^x$ for $\{x, y\} \notin F$ and $(L')_y^x = 0$ for $\{x, y\} \in F$. To be more precise,

$$L' = \begin{bmatrix} L|_{S_1 \times S_1} & 0 & \cdots & 0 \\ 0 & L|_{S_2 \times S_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L|_{S_k \times S_k} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbb{P}[\mathcal{C}_\alpha \text{ is finer than } \pi] &= \left(\frac{\det(L')}{\det(L)} \right)^{-\alpha} = (\det((-L')^{-1}(-L)))^\alpha \\ &= \left(\det \left(- \begin{bmatrix} V^{S_1} & 0 & \cdots & 0 \\ 0 & V^{S_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V^{S_k} \end{bmatrix} L \right) \right)^\alpha = \left| \begin{array}{cccc} I & -H_{1,2} & \cdots & -H_{1,k} \\ -H_{2,1} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & -H_{k-1,k} \\ -H_{k,1} & \cdots & -H_{k,k-1} & I \end{array} \right|^\alpha \\ &= (\det(I - K))^\alpha. \end{aligned}$$

(Note that $V^{S_i} L|_{S_i \times S_j} = H_{S_i, S_j}^{S_i^c} = H_{i,j}$ for $i \neq j \in \{1, \dots, k\}$.) □

4.8 An example on the discrete circle

Consider a discrete circle G with n vertices $1, \dots, n$ and $2n$ oriented edges

$$E = \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1), (2, 1), (3, 2), \dots, (n, n-1), (1, n)\}$$

Define the clockwise edges set $E_+ = \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$ and the counter clockwise edges $E_- = E - E_+$. Consider a Markovian generator L such that for any $e \in E_+$, $L_{e+}^{e-} = p$, $L_{e-}^{e+} = 1 - p$, $L_{e-}^{e-} = -(1 + c)$ and L is null elsewhere. Then, we have a loop measure and Poissonian ensembles associated with L . The rest of this subsection is devoted to study the loop cluster \mathcal{C}_α in this example.

Lemma 4.21. *Let $T_{3,n}$ be a $n \times n$ tri-diagonal Toeplitz matrix of the following form:*

$$\begin{bmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c & a & b \\ 0 & \cdots & 0 & c & a \end{bmatrix}_{n \times n}.$$

Let S_n be the following $n \times n$ matrix:

$$\begin{bmatrix} a & b & 0 & & c \\ c & a & b & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & c & a & b \\ b & & 0 & c & a \end{bmatrix}_{n \times n}.$$

Let x_1, x_2 be the roots of $x^2 - ax + bc = 0$. Then,

- $\det(T_{3,n}) = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2},$
- $\det(S_n) = x_1^n + x_2^n + (-1)^{n+1}(b^n + c^n).$

Proposition 4.22.

$$\begin{aligned} \text{Set } x_1 &= \frac{1}{2}(1 + c + \sqrt{(1+c)^2 - 4p(1-p)}), \\ x_2 &= \frac{1}{2}(1 + c - \sqrt{(1+c)^2 - 4p(1-p)}). \end{aligned}$$

Then,

$$\mathbb{P}[\{1, n\} \text{ is closed}] = \left(\frac{(x_1^n - x_2^n)^2}{(x_1 - x_2)(x_1^{n-1} - x_2^{n-1})(x_1^n + x_2^n - (p^n + (1-p)^n))} \right)^{-\alpha}.$$

Proof. By Proposition 3.9 and Proposition 3.34

$$\begin{aligned} \mathbb{P}[\{1, n\} \text{ is closed}] &= e^{-\alpha\mu(N_n^1(l) + N_1^n(l) > 0)} = e^{-\alpha\mu(l \text{ visits } 1 \text{ and } n)} \\ &= \left(\frac{\det(V_{\{1,n\}})}{V_1^1 V_n^n} \right)^\alpha = \left(\frac{\det(-L|_{\{2,\dots,n\} \times \{2,\dots,n\}}) \det(L|_{\{1,\dots,n-1\} \times \{1,\dots,n-1\}})}{\det(-L|_{\{2,\dots,n-1\} \times \{2,\dots,n-1\}}) \det(-L)} \right)^{-\alpha} \\ &= \left(\frac{(x_1^n - x_2^n)^2}{(x_1 - x_2)(x_1^{n-1} - x_2^{n-1})(x_1^n + x_2^n - (p^n + (1-p)^n))} \right)^{-\alpha} \end{aligned}$$

where $x_1 = \frac{1+c+\sqrt{(1+c)^2-4p(1-p)}}{2}$ and $x_2 = \frac{1+c-\sqrt{(1+c)^2-4p(1-p)}}{2}$. □

Proposition 4.23. *Conditionally on $\{1, n\}$ being closed, \mathcal{C}_α is a renewal process conditioned to jump at time n . To be more precise, by deleting edges $\{1, n\}$ and adding $\{0, 1\}, \{n, n+1\}$, we get a discrete segment with vertices $\{0, 1, \dots, n, n+1\}$ and edges $\{\{0, 1\}, \dots, \{n, n+1\}\}$. Conditionally to $\{1, n\}$ being closed, \mathcal{C}_α induces a partition on $\{1, \dots, n\}$. The clusters of \mathcal{C}_α are the intervals between the edges closed at time α (namely the edges which are not crossed by any loop of \mathcal{L}_α). Then the left points of these closed edges, together with the left points of $\{0, 1\}$ and $\{n, n+1\}$, form a renewal process conditioned to jump at n .*

Proof. Among the Poissonian loop ensembles, the ensemble of loops crossing $\{1, n\}$ and the rest are independent. Therefore, the conditional law \mathcal{Q} of the loops not crossing $\{1, n\}$ conditioned on the event that no loop is crossing $\{1, n\}$ is exactly the same as the unconditioned law. Consider another Poissonian loop ensembles on \mathbb{Z} driven by the following generator:

$$L_m^m = -(1+c), L_{m+1}^m = p, L_{m-1}^m = 1-p \text{ for all } m \in \mathbb{Z}, \text{ and } L \text{ is null elsewhere.}$$

Then, \mathcal{Q} is the same as the conditional law of the loop ensembles contained in $\{1, \dots, n\}$ given the condition that $\{0, 1\}$ or $\{n, n+1\}$ are closed. By Proposition 3.2, after a harmonic transform, L is modified as follows:

$$L_m^m = -(1+c), L_{m+1}^m = L_{m-1}^m = \sqrt{p(1-p)} \text{ for all } m \in \mathbb{Z}, \text{ and } L \text{ is null elsewhere.}$$

According to Proposition 3.1 in [LJL12], in the case of \mathbb{Z} , conditionally to the event that $\{0, 1\}$ is closed, the left points of the closed edges form a renewal process. There is an obvious one-to-one correspondence between the jumps of the renewal process and the closed edges. Finally, in the case of the circle, conditioning on $\{1, n\}$ being closed, we can identify \mathcal{C}_α to a renewal process conditioned to jump at time n . It is not hard to see the parameter κ in [LJL12] equals $\frac{1+c-2\sqrt{p(1-p)}}{\sqrt{p(1-p)}}$. \square

5 Loop erasure and spanning tree

In this section we will show that Poisson processes of loops appear naturally in the construction of random spanning trees.

5.1 Loop erasure

Suppose ω is the path of a minimal transient canonical Markov process, then its path can be expressed as a sequence $(x_0, t_0, x_1, t_1, \dots)$. The corresponding discrete path (x_0, x_1, \dots) is the embedded Markov chain. From the transience assumption, $\sum_{n \in \mathbb{N}} 1_{\{x_n = x\}} < \infty$ a.s..

Definition 5.1 (Loop erasure). The loop erasure operation which maps a path ω to its loop erased path ω_{BE} is defined as: $\omega_{BE} = (y_0, \dots)$ with $y_0 = x_0$. Define $T_0 = \inf\{n \in \mathbb{N} : \forall m \geq n, x_m \neq y_0\}$, then set $y_1 = x_{T_0}$. Similarly define $T_1 = \inf\{n \in \mathbb{N} : \forall m \geq n, x_m \neq y_1\}$, set $y_2 = x_{T_1}$ and so on. Let \mathbb{P}_{BE}^ν be the image measure of \mathbb{P}^ν where ν is the initial distribution of the Markov process.

Recall that ∂ is the cemetery point, that $Q_\partial^x = 1 - \sum_{y \neq \partial} Q_y^x$ for $x \neq \partial$ and $Q_x^\partial = \delta_x^\partial$. Set $L_\partial^x = -\sum_{y \neq \partial} L_y^x$ for $x \neq \partial$, $L_\partial^\partial = -1$ and $L_x^\partial = 0$ for $x \neq \partial$.

Proposition 5.1. *We have the following finite marginal distribution for the loop-erased random walk:*

$$\begin{aligned} \mathbb{P}_{BE}^\nu[\omega_{BE} = (x_0, x_1, \dots, x_n, \dots)] \\ = \nu_{x_0} \det(V_{\{x_0, \dots, x_{n-1}\}}) L_{x_1}^{x_0} \dots L_{x_n}^{x_{n-1}} \mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty] \\ = \nu_{x_0} L_{x_1}^{x_0} \dots L_{x_n}^{x_{n-1}} \begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \dots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}. \end{aligned}$$

Proof. Starting from x_n , the probability that the Markov process never reaches the set $\{x_0, \dots, x_{n-1}\}$, $\mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty]$ equals the same probability for the trace of the Markov process on x_0, \dots, x_n , $\mathbb{P}_{\{x_0, \dots, x_n\}}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty]$. It equals the one step transition probability from x_n to ∂ for the trace of the process. Let $L_{\{x_0, \dots, x_n\}}$ be the generator of the trace of the Markov process on $\{x_0, \dots, x_n\}$. Then, the one step transition probability from x_n to ∂ equals $\frac{(L_{\{x_0, \dots, x_n\}})_{\partial}^{x_n}}{-(L_{\{x_0, \dots, x_n\}})_{x_n}^{x_n}}$. Since $(L_{\{x_0, \dots, x_n\}})_{\partial}^{x_n} = -(L_{\{x_0, \dots, x_n\}})_{x_n}^{x_n} - \sum_{i=0}^{n-1} (L_{\{x_0, \dots, x_n\}})_{x_i}^{x_n}$ and $-(L_{\{x_0, \dots, x_n\}})_{x_i}^{x_n} = (-1)^{i+1+n+1} \frac{\det(V|_{\{x_0, \dots, x_n\} \setminus \{x_i\} \times \{x_0, \dots, x_{n-1}\}})}{\det(V_{\{x_0, \dots, x_n\}})}$, we have

$$(L_{\{x_0, \dots, x_n\}})_{\partial}^{x_n} = \frac{1}{\det(V_{\{x_0, \dots, x_n\}})} \begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \dots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}$$

$$\text{and } -(L_{\{x_0, \dots, x_n\}})_{x_n}^{x_n} = \begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} \end{vmatrix}.$$

$$\text{Therefore, } \mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty] = \frac{\begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \dots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}}{\begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} \end{vmatrix}}.$$

Set $D_0 = \phi$ and $D_k = \{x_0, \dots, x_{k-1}\}$ for $k \in \mathbb{N}_+$. Note that $Q|_{D_k^c \times D_k^c}$ is the transition probability for the process restricted in D_k^c . In order for the loop-erased path ω_{BE} to be $(x_0, x_1, \dots, x_n, \dots)$, the random walk must start from x_0 . After some excursions back to x_0 , it should jump to x_1 and never return to x_0 . Next, after some excursions from x_1 to x_1 , it jumps to x_2 and never returns to x_0, x_1 , etc. Accordingly,

$$\begin{aligned} \mathbb{P}_{BE}^\nu[\omega_{BE} = (x_0, x_1, \dots, x_n, \dots)] \\ = \nu_{x_0} \prod_{k=0}^{n-1} \left(\sum_{n \geq 0} ((Q|_{D_k^c \times D_k^c})^n)_{x_k}^{x_k} Q_{x_{k+1}}^{x_k} \mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty] \right) \\ = \nu_{x_0} \prod_{k=0}^{n-1} (V_{x_k}^{D_k^c})_{x_k}^{x_k} L_{x_{k+1}}^{x_k} \mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty] \end{aligned}$$

where $L^{D_k^c} = L|_{D_k^c \times D_k^c}$ is the generator of the Markov process restricted in D_k^c , and $V^{D_k^c}$ be the corresponding potential, see Definition 2.4.

Let V_F stands for the sub-matrix of V restricted to $F \times F$. It is also the potential of the trace of the Markov process on F and let \mathbb{P}_F stand for its law. Then, for all $D \subset F$, we have $(V^{D^c})_F = (V_F)^{D^c}$. In particular, for $k < n$, we have $(V^{D_k^c})_{x_k}^{x_k} = ((V^{D_k^c})_{D_n})_{x_k}^{x_k} = ((V_{D_n})^{D_k^c})_{x_k}^{x_k}$. One can apply Jacobi's formula

$$\det(A|_{B \times B}) \det(A^{-1}) = \det(A^{-1}|_{B^c \times B^c})$$

for $A = (V_{D_n})^{D_k^c}$ and $B = \{x_k\}$. To be more precise, since $((V_{D_n})^{D_k^c})^{-1} = (-L_{D_n})|_{D_k^c \times D_k^c} = (-L_{D_n})|_{(D_n - D_k) \times (D_n - D_k)}$, we have

$$(V^{D_k^c})_{x_k}^{x_k} = ((V_{D_n})^{D_k^c})_{x_k}^{x_k} = \frac{\det(-L_{D_n}|_{(D_n - D_{k+1}) \times (D_n - D_{k+1})})}{\det(-L_{D_n}|_{(D_n - D_k) \times (D_n - D_k)})}$$

with the convention that $\det(-L_{D_n}|_\phi) = 1$. Then,

$$\begin{aligned} \prod_{k=0}^{n-1} (V^{D_k^c})_{x_k}^{x_k} &= \prod_{k=0}^{n-1} \frac{\det(-L_{D_n}|_{(D_n - D_{k+1}) \times (D_n - D_{k+1})})}{\det(-L_{D_n}|_{(D_n - D_k) \times (D_n - D_k)})} = \frac{\det(-L_{D_n}|_{(D_n - D_n) \times (D_n - D_n)})}{\det(-L_{D_n}|_{(D_n - D_0) \times (D_n - D_0)})} \\ &= \frac{1}{\det(-L_{D_n})} = \det((V_{D_n})_{D_n}) = \det(V_{\{x_0, \dots, x_{n-1}\}}). \end{aligned}$$

Finally, by combining the results above, we conclude that

$$\begin{aligned} \mathbb{P}_{BE}^\nu[\omega_{BE} = (x_0, x_1, \dots, x_n, \dots)] \\ = \nu_{x_0} \det(V_{\{x_0, \dots, x_{n-1}\}}) L_{x_1}^{x_0} \dots L_{x_n}^{x_{n-1}} \mathbb{P}^{x_n}[T_{\{x_0, \dots, x_{n-1}\}} = \infty] \\ = \nu_{x_0} L_{x_1}^{x_0} \dots L_{x_n}^{x_{n-1}} \begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_{n-1}}^{x_0} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ V_{x_0}^{x_{n-1}} & \dots & V_{x_{n-1}}^{x_{n-1}} & 1 \\ V_{x_0}^{x_n} & \dots & V_{x_{n-1}}^{x_n} & 1 \end{vmatrix}. \end{aligned}$$

□

Remark 13. Since a Markov chain in a countable space could be viewed as a pure-jump sub-Markov process with jumping rate 1, the above result holds for a sub-Markov chain if we replace L by the transition matrix $Q - Id$ and $V = (Id - Q)^{-1}$.

The following property was discovered by Omer Angel and Gady Kozma, see Lemma 1.2 in [Koz07]. Here, we give a different proof as an application of Proposition 5.1.

Proposition 5.2. *Let $(X_m, m \in [0, \zeta])$ be a discrete Markov chain in a countable space S with time life ζ and initial point x_0 . Fix some $w \in S \setminus \{x_0\}$, define $T_1 = \inf\{n > 0 : X_n = w\}$ and $T_N = \inf\{m > T_{N-1} : X_m = w\}$ with the convention that $\inf \phi = \infty$. We can perform loop-erasure for the path (X_0, \dots, X_{T_N}) , and let $LE[0, T_N]$ stand for the loop-erased self-avoiding path obtained in that way. If $T_N < \infty$ with positive probability, then the conditional law of $LE[0, T_N]$ given that $\{T_N < \infty\}$ is the same as the conditional law of $LE[0, T_1]$ given that $\{T_1 < \infty\}$.*

Proof. We suppose $T_1 < \infty$ with positive probability. By adding a small killing rate ϵ at all states and taking $\epsilon \downarrow 0$, we could suppose that we have a positive probability to jump to the cemetery point from any state. In particular, the Markov chain is transient.

Let ∂ be the cemetery point. Let $\tau(p)$ be a geometric variable with mean $1/p$, independent of the Markov chain. Let $(X_m^{(p)}, m \in [0, (\zeta - 1) \wedge T_{\tau(p)}])$ be the sub-Markov chain X stopped after $T_{\tau(p)}$ which is again sub-Markov. Let $\mathbb{P}_p^{x_0}$ stand for the law of $X^{(p)}$ and let $\mathbb{P}_{p, BE}^{x_0}$ stand for the law of the loop-erased random walk associated to $(X_m^{(p)}, m \in [0, (\zeta - 1) \wedge T_{\tau(p)}])$. Let $Q^{(p)}$ be the transition matrix of $X^{(p)}$ and use the notation Q for $Q^{(0)}$. Then, $(Q^{(p)})_i^w = (1 - p)Q_i^w$ for $i \in S$ and $(Q^{(p)})_j^i = Q_j^i$ for $i \in S \setminus \{w\}$ and $j \in S$. Accordingly, $(Q^{(p)})_\partial^w = p + Q_\partial^w - pQ_\partial^w$. Define $V = (I - Q)^{-1}$, $V_{q\delta_w} = (M_{q\delta_w} + I - Q)^{-1}$ for $q \geq 0$ and $V^{(p)} = (I - Q^{(p)})^{-1} = (M_{(1-p)\delta_w}(I + \frac{p}{1-p} - Q))^{-1} = V_{\frac{p}{1-p}\delta_w} M_{\frac{1}{1-p}\delta_w}$. Set $C_w = \{\text{the loop-erased random walk stopped at } w\}$. Then,

$$\begin{aligned} C_\Omega &= \{\text{the random walk stopped at } w\} \\ &= \bigcup_{n \geq 1} \{\text{the random walk stopped at } w \text{ at time } T_n\} \\ &= \bigcup_{k \geq 1} \{\tau(p) = k, T_k < \zeta\} \bigcup_{k \geq 1} \{T_k = \zeta - 1, \tau(p) > k\}. \end{aligned}$$

For $x_n = w$,

$$\begin{aligned} \mathbb{P}_{p, BE}^{x_0}[\omega_{BE} = (x_0, x_1, \dots, x_n = w)] \\ = (Q^{(p)})_{x_1}^{x_0} \dots (Q^{(p)})_{x_n}^{x_{n-1}} (Q^{(p)})_\partial^{x_n} \begin{vmatrix} (V^{(p)})_{x_0}^{x_0} & \dots & (V^{(p)})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V^{(p)})_{x_0}^{x_n} & \dots & (V^{(p)})_{x_n}^{x_n} \end{vmatrix} \end{aligned}$$

$$= \frac{p + Q_\partial^w - pQ_\partial^w}{(1-p)Q_\partial^w} Q_{x_1}^{x_0} \dots Q_{x_n}^{x_{n-1}} Q_\partial^{x_n} \begin{vmatrix} (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix}.$$

By the resolvent equation, $V_j^i = (V_{\frac{p}{1-p}\delta_w})_j^i + \frac{p}{1-p} (V_{\frac{p}{1-p}\delta_w})_w^i V_j^w = (V_{\frac{p}{1-p}\delta_w})_j^i + \frac{p}{1-p} (V_{\frac{p}{1-p}\delta_w})_j^w V_w^i$.
Therefore,

$$\begin{aligned} \begin{vmatrix} V_{x_0}^{x_0} & \dots & V_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ V_{x_0}^{x_n} & \dots & V_{x_n}^{x_n} \end{vmatrix} &= \begin{vmatrix} 1 & -\frac{p}{1-p} V_{x_0}^{x_n} & \dots & -\frac{p}{1-p} V_{x_n}^{x_n} \\ (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix} \\ &= \begin{vmatrix} 1 + \frac{p}{1-p} V_{x_n}^{x_n} & -\frac{p}{1-p} V_{x_0}^{x_n} & \dots & -\frac{p}{1-p} V_{x_n}^{x_n} \\ 0 & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix} \\ &= (1 + \frac{p}{1-p} V_{x_n}^{x_n}) \begin{vmatrix} (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_0} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_0} \\ \vdots & \ddots & \vdots \\ (V_{\frac{p}{1-p}\delta_w})_{x_0}^{x_n} & \dots & (V_{\frac{p}{1-p}\delta_w})_{x_n}^{x_n} \end{vmatrix}. \end{aligned}$$

Accordingly, $\frac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w} \mathbb{P}_{p,BE}^{x_0}[\omega_{BE} = (x_0, x_1, \dots, x_n = w)]$ does not depend on p .
Consequently, it must be equal to $\mathbb{P}_{0,BE}^{x_0}[\omega_{BE} = (x_0, x_1, \dots, x_n = w)]$. Equivalently,

$$\frac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w} \mathbb{P}_{p,BE}^{x_0}[\cdot, C_w] = \mathbb{P}_{0,BE}^{x_0}[\cdot, C_w]$$

Therefore,

$$\mathbb{P}_{0,BE}^{x_0}[C_w] = \frac{(1-p+pV_w^w)Q_\partial^w}{p+Q_\partial^w-pQ_\partial^w} \mathbb{P}_{p,BE}^{x_0}[C_w].$$

Immediately, it implies that conditionally on C_w , the law of the loop-erased random walk does not depend on p :

$$\mathbb{P}_{p,BE}^{x_0}[\cdot | C_w] = \mathbb{P}_{0,BE}^{x_0}[\cdot | C_w].$$

Since

$$\begin{aligned} \mathbb{P}_{p,BE}^{x_0}[\omega_{BE} \in \cdot, C_w] &= \sum_{k \geq 1} \mathbb{P}^{x_0}[\tau(p) = k, T_k < \zeta, LE[0, T_k] \in \cdot] \\ &\quad + \sum_{k \geq 1} \mathbb{P}^{x_0}[\tau(p) > k, T_k = \zeta - 1, LE[0, T_k] \in \cdot] \\ &= \sum_{k \geq 1} (1-p)^{k-1} p \mathbb{P}^{x_0}[T_k < \infty, LE[0, T_k] \in \cdot] \\ &\quad + \sum_{k \geq 1} (1-p)^k \mathbb{P}^{x_0}[T_k < \infty, LE[0, T_k] \in \cdot] Q_\partial^w \end{aligned}$$

$$= \sum_{k \geq 1} (1-p)^{k-1} (p + Q_\partial^w - pQ_\partial^w) \mathbb{P}^{x_0}[T_k < \infty] \mathbb{P}^{x_0}[LE[0, T_k] \in \cdot | T_k < \infty],$$

we have

$$\mathbb{P}_{p, BE}^{x_0}[\omega_{BE} \in \cdot | C_w] = \frac{\sum_{k \geq 1} (1-p)^{k-1} \mathbb{P}^{x_0}[T_k < \infty] \mathbb{P}^{x_0}[LE[0, T_k] \in \cdot | T_k < \infty]}{\sum_{k \geq 1} (1-p)^{k-1} \mathbb{P}^{x_0}[T_k < \infty]}. \quad (*)$$

Since $\mathbb{P}_{p, BE}^{x_0}[\omega_{BE} \in \cdot | C_w]$ does not depend on $p \in [0, 1]$, we will denote it by \mathbb{Q} . Then the equation (*) can be written as follows:

$$\begin{aligned} \mathbb{Q}[\cdot] \sum_{k \geq 1} (1-p)^{k-1} \mathbb{P}^{x_0}[T_k < \infty] \\ = \sum_{k \geq 1} (1-p)^{k-1} \mathbb{P}^{x_0}[T_k < \infty] \mathbb{P}^{x_0}[LE[0, T_k] \in \cdot | T_k < \infty]. \end{aligned}$$

Finally, by identifying the coefficients, we conclude that $\mathbb{P}^{x_0}[LE[0, T_k] \in \cdot | T_k < \infty] = \mathbb{Q}[\cdot]$ as long as $\mathbb{P}^{x_0}[T_k < \infty]$ for $k \geq 1$ and we are done. \square

Consider $(e_t, t \geq 0)$, a Poisson point process of excursions of finite lifetime at x with the intensity $Leb \otimes (-L_x^x - \frac{1}{V_x^x}) \nu_{\{x\}, ex}^{x \rightarrow x}$. (Recall that $\nu_{\{x\}, ex}^{x \rightarrow x}$ is the normalized excursion measure at x , see Definition 3.16.) Let $(\gamma(t), t \geq 0)$ be an independent Gamma subordinator⁵ with the Laplace exponent

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) s^{-1} e^{-s/V_x^x} ds.$$

Let \mathfrak{R}_α be the closure of the image of the subordinator γ up to time α , i.e. $\mathfrak{R}_\alpha = \overline{\{\gamma(t) : t \in [0, \alpha]\}}$. Then, $[0, \gamma(\alpha)] \setminus \mathfrak{R}_\alpha$ is the union of countable disjoint open intervals, $\{] \gamma(t-), \gamma(t) [: t \in [0, \alpha], \gamma(t-) < \gamma(t)\}$. To such an open interval $]g, d[$, one can associate a based loop l as follows: During the time interval $]g, d[$, the Poisson point process $(e_t, t \geq 0)$ has finitely many excursions, namely $e_{t_1}, \dots, e_{t_n}, g < t_1 < \dots < t_n < d$. Each excursion e_{t_i} is viewed as a càdlàg path of lifetime ζ_{t_i} : $(e_{t_i}(s), s \in [0, \zeta_{t_i}[)$. Define $l : [0, d - g + \sum_i \zeta_{t_i}] \rightarrow S$ as follows:

$$l(s) = \begin{cases} e_{t_i}(s - (\sum_{j < i} \zeta_{t_j} + t_i)) & \text{if } s \in [\sum_{j < i} \zeta_{t_j} + t_i, \sum_{j \leq i} \zeta_{t_j} + t_i[\\ x & \text{otherwise.} \end{cases}$$

This mapping between an open interval $]g, d[$ and a based loop l depends on $]g, d[$ and $(e_t, t \in]g, d[)$ and we denote it by $\Psi^{]g, d[}$ ($l = \Psi^{]g, d[}(e)$). By mapping a based loop into a loop, we get a countable collection of loops for $\alpha \geq 0$, namely \mathcal{O}_α .

Proposition 5.3. *($\mathcal{O}_\alpha, \alpha \geq 0$) has the same law as the Poisson point process of loops intersecting $\{x\}$, i.e. $(\{l \in \mathcal{L}_\alpha : l^x > 0\}, \alpha > 0)$.*

⁵See Chapter III of [Ber96].

Proof. As both sides have independent stationary increment, it is enough to show $\mathcal{O}_1 = \{l \in \mathcal{L}_1 : l^x > 0\}$. It is well-known that $(\gamma(t) - \gamma(t-), t \in \mathbb{R})$ is a Poisson point process with characteristic measure $\frac{1}{s}e^{-s/V_x^x} ds$. Therefore, $\sum_{l \in \mathcal{O}_\alpha} \delta_{l^x}$ is Poisson random measure with intensity $\frac{1}{s}e^{-s/V_x^x} ds$. On the other hand, for the Poisson ensemble of loops \mathcal{L}_α , by taking the trace of the loops on x and dropping the empty ones, as a consequence of Proposition 3.9, we get a Poisson ensemble of trivial Markovian loops with intensity measure $\frac{1}{s}e^{s(L_{\{x\}}^x)_x} ds$ where $(-L_{\{x\}})_x = 1/V_x^x$. Consequently, we have

$$\{l^x : l \in \mathcal{O}_1\} \text{ has the same law as } \{l^x : l \in \mathcal{L}_1, l^x > 0\}$$

In other words, by disregarding the excursions attached to each loop, the set of trivial loops in x obtained from \mathcal{O}_1 and $\{l \in \mathcal{L}_1 : l^x > 0\}$ is the same. In order to restore the loops, we need to insert the excursions into the trivial loops. Then, it remains to show that the excursions are inserted into the trivial loops in the same way. Finally, by using the independence between $(e_t, t \geq 0)$ and $(\gamma(t), t \geq 0)$ and the stationary independent increments property with respect to time t , it ends up in proving the following affirmation: $\Psi^{[0,T[}(e)$ induces the same probability on the loops with $l^x = T$ as the loop measure conditioned by $\{l^x = T\}$. By Proposition 3.7, we have $l^x \mu(dl) = \mu^{x,x}(dl)$. Hence, $\mu(dl|l^x = T) = \mu^{x,x}(dl|l^x = T)$ where $\mu^{x,x}$ is considered to be a loop measure. Let \mathbb{P}^x be the law of the Markov process $(X_t, t \in [0, \zeta[)$ associated with the Markovian loop measure μ . Let $(L(x, t), t \in [0, \zeta[)$ be the local time process at x and $L^{-1}(x, t)$ be its right-continuous inverse. Let τ be an independent exponential variable with parameter 1. Define the process $X^{L^{-1}(x, \tau)}$ with lifetime $L^{-1}(x, \tau) \wedge \zeta$ as follows: $X^{L^{-1}(x, \tau)}(T) = X(T), T \in [0, L^{-1}(x, \tau) \wedge \zeta[$. Denote by $\mathbb{Q}[dl]$ the law of $X^{L^{-1}(x, \tau)}$. Then, $e^{-l^x} \mu^{x,x}(dl) = \mathbb{Q}[dl]$ where $\mu^{x,x}(dl)$ is considered to be a based loop measure. Therefore,

$$\mu^{x,x}(dl|l^x = T) = \mathbb{Q}[dl|l^x = T] = \mathbb{Q}[dl|\tau = T] = \text{the law of } \Psi^{[0,T[}(e)$$

in the sense of based loop measures. Then, the equality stills holds for loop measures and we are done. \square

Suppose $(X_t, t \in [0, \zeta[)$ is a transient Markov process on S . (Assume the process stays at the cemetery point after lifetime ζ .) Define the local time at x $L(x, t) = \int_0^t 1_{\{X_s=x\}} ds$. Denote by $L^{-1}(x, t)$ its right-continuous inverse and by $L^{-1}(x, t-)$ its left-continuous inverse. The excursion process $(e_t, t \geq 0)$ is defined by $e_t(s) = X_{s+L^{-1}(x, t-)}, s \in [0, L^{-1}(x, t) - L^{-1}(x, t-)[$. Define a measure on the excursion which never returns to x by

$$\tilde{\nu}^{x \rightarrow}[dl] = \sum_{y \in S} Q_y^x \mathbb{P}^y[dl, \text{the process never hits } x].$$

We can calculate the total mass of $\tilde{\nu}^{x \rightarrow}$ as follows:

$$\begin{aligned}\tilde{\nu}^{x \rightarrow}[1] &= \sum_{y \in S} Q_y^x \mathbb{P}^y[\text{the process never hits } x] \\ &= 1 - \mathbb{E}^x[\{\text{after leaving } x, \text{ the process returns to } x\}] \\ &= (1 - (R^{\{x\}})^x) = \frac{(L_{\{x\}}^x)_x}{L_x^x} = -\frac{1}{V_x^x L_x^x}.\end{aligned}$$

After normalization, we get a probability measure $\nu^{x \rightarrow}$. The law of the first excursion is $-\frac{1}{V_x^x L_x^x} \nu^{x \rightarrow} + \left(1 + \frac{1}{V_x^x L_x^x}\right) \nu_{\{x\}, ex}^{x \rightarrow x}$. In particular, the first excursion is not an excursion from x back to x with probability $-\frac{1}{L_x^x V_x^x}$. According to the excursion theory, the excursion process is a Poisson point process stopped at the appearing of an excursion of infinite lifetime or an excursion that ends up at the cemetery. The characteristic measure is proportional to the law of the first excursion. By taking the trace of the process on x , we know that the total occupation time is an exponential variable with parameter $(-L_{\{x\}}^x)_x = \frac{1}{V_x^x}$. According to the excursion theory, it is an exponential variable with parameter $\frac{-d}{V_x^x L_x^x}$, d being the mass of the characteristic measure. Immediately, we get $d = -L_x^x$. If we focus on the process of excursions from x back to x , it is a Poisson point process with characteristic measure $(-L_x^x - \frac{1}{V_x^x}) \nu_{\{x\}, ex}^{x \rightarrow x}$ stopped at an independent exponential time with parameter $\frac{1}{V_x^x}$. Let $(\gamma(t), t \geq 0)$ be a Gamma subordinator with Laplace exponent

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) s^{-1} e^{-s/V_x^x} ds.$$

Then, $\gamma(t)$ follows the $\Gamma(t, \frac{1}{V_x^x})$ distribution with density $\rho(y) = \frac{1}{\Gamma(t)(V_x^x)^t} y^{t-1} e^{-y/V_x^x}$. In particular, $\gamma(1)$ is an exponential variable of the parameter $1/V_x^x$. It is known that $(\frac{\gamma(t)}{\gamma(1)}, t \in [0, 1])$ is independent of $\gamma(1)$, and that it is a Dirichlet process. (One can prove this by a direct calculation on the finite marginal distribution.) Moreover, the jumps of the process $(\frac{\gamma(t)}{\gamma(1)}, t \in [0, 1])$ rearranged in decreasing order follow the Poisson-Dirichlet $(0, 1)$ distribution. For $x \in S$, let Z_x be the last passage time in x : $Z_x = \sup\{t \in [0, \zeta[: X(t) = x\}$. Suppose the loop erased path ω_{BE} equals (x_1, \dots) . Define $S_n = T_{x_n}$ for $n \geq 1$ and $S_0 = 0$. Let O_i be $(X_s, s \in [S_i, S_{i+1}[[$ i.e. the i -th loop erased from the process X . Then O_1 can be viewed as a Poisson point process $(e_t^{(1)}, t \in [0, L(x_1, \zeta)[[$) of excursions at x_1 killed at the arrival of an excursion with infinite lifetime or an excursion ending up at the cemetery. Conditionally on $\omega_{BE} = (x_1, x_2, \dots)$, the shifted process $(X(s + T_1), s \in [0, \zeta[)$ is the Markov process restricted in $S \setminus \{x_1\}$ starting from $x_2 = X(T_1)$. Moreover, it is conditionally independent of the Poisson point process $e^{(1)}$. Once again, we can view O_2 as an killed Poisson point process of excursions at x_2 and denote it by $e^{(2)}$. Clearly, we have the independence between $e^{(1)}$ and $e^{(2)}$ conditionally on $\omega_{BE} = (x_1, x_2, \dots)$. Repeating

this procedure, we get a sequence of point process of excursions $e^{(1)}, \dots$. Conditionally on $\omega_{BE} = (x_1, \dots, x_n, \dots)$, they are independent, and $e^{(n)}$ has the same law as the killed excursion process for the Markov process restricted in $D_n = S \setminus \{x_1, \dots, x_{n-1}\}$. Let $O_i^{x_i}$ be the occupation time at x_i for the based loop O_i . Let $(\gamma_t^{(i)}, t \geq 0), i \geq 1$ be a sequence of independent Gamma subordinators with Laplace exponent

$$\Phi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) s^{-1} \exp\left(-\frac{s}{(V^{D_n})_{x_i}^{x_i}}\right) ds.$$

We suppose they are independent of the Markov process. Then, $O_i^{x_i}, i \geq 1$ has the same law as $\gamma^{(i)}(1), i \geq 1$ conditionally on ω_{BE} . In the spirit of Proposition 5.3 by cutting the excursion process according to the range of subordinator, if at time $\alpha \in [0, 1]$, we cut the loop O_i according to the range of $(\frac{\gamma^{(i)}(s)O_i^{x_i}}{\gamma^{(i)}(1)}, s \in [0, \alpha])$, we get a point process of loops $(\mathcal{O}_\alpha^{(i)}, \alpha \in [0, 1])$. Conditionally on ω_{BE} , it has the same law as the Poisson point process $(\mathcal{L}_\alpha^{D_i} \setminus \mathcal{L}_\alpha^{D_{i+1}}, \alpha \in [0, 1])$. Moreover, conditionally on ω_{BE} , $(\mathcal{O}_\alpha^{(i)}, \alpha \in [0, 1]), i \geq 1$ are independent. The definition of the Poisson random measure ensures independence among $(\mathcal{L}_\alpha^{D_i} \setminus \mathcal{L}_\alpha^{D_{i+1}}, \alpha \in [0, 1]), i = 1, \dots$. Consequently, we have the following proposition.

Proposition 5.4. *Conditionally on ω_{BE} , $(\mathcal{O}_\alpha, \alpha \in [0, 1])$ has the same law as $(\{l \in \mathcal{L}_\alpha : l \text{ intersects } \omega_{BE}\}, \alpha \in [0, 1])$.*

Remark 14. The jumps of the process $\frac{\gamma(t)}{\gamma(1)}$ rearranged in decreasing order follow the Poisson-Dirichlet $(0, 1)$ distribution. Since a Poisson point process is always homogeneous in time, the following two cutting method gives the same loop ensemble in law:

- Cutting the loop according to the range of $\left(\frac{\gamma(t)}{\gamma(1)}, t \in [0, 1]\right)$,
- Cutting the loop according to the Poisson-Dirichlet $(0, 1)$ distribution.

As a result, a similar result holds for $\alpha = 1$ if we cut the loops according to the Poisson-Dirichlet $(0, 1)$ distribution.

5.2 Random spanning tree

Throughout this section, we consider a finite state space S with a transient Markov process $(X_t, t \geq 0)$ on it. Denote by Δ the cemetery point for X . As usual, denote by L the generator of X and by Q the transition matrix of the embedded Markov chain.

By the following algorithm, one can construct a random spanning tree of $S \cup \{\Delta\}$ rooted⁶ at Δ . We give an orientation on the tree: each edge is directed towards the root.

⁶By a random spanning tree rooted at Δ , we mean a random spanning tree with a special mark on the vertex Δ .

Definition 5.2 (Wilson's algorithm). Choose an arbitrary order on S : $S = \{v_1, \dots, v_n\}$. Define $S_0 = \{\Delta\}$. Let T_0 be the tree with single vertex Δ . We recurrently construct a series of growing random trees $T_k, k \in \mathbb{N}$ as follows:

Suppose T_k is well-constructed with set of vertices S_k . If $S \cup \{\Delta\} \setminus S_k = \emptyset$, then we stop the procedure and set $\mathcal{T} = T_k$. Otherwise, there is a unique vertex in $S \cup \{\Delta\} \setminus S_k$ with the smallest sub-index and we denote it by y_{k+1} . Run a Markov chain from y_{k+1} with transition matrix Q . It will hit S_k in finitely many steps. We stop the Markov chain after it reaches S_k and erase progressively the loops according to the Definition 5.1. In this way, we get a loop-erased path η_{k+1} joining y_{k+1} to T_k . By adding this loop erased path η_{k+1} to T_k , we construct the random tree T_{k+1} . The procedure will stop after a finite number of steps and it produces a random spanning tree \mathcal{T} .

Proposition 5.5. Denote by $\mu_{ST,\Delta}$ the distribution of the random spanning tree rooted at Δ given by Wilson's algorithm. Then,

$$\mu_{ST,\Delta}(\mathcal{T} = T) = \det(V) 1_{\{T \text{ is a spanning tree rooted at } \Delta\}} \prod_{\substack{(x,y) \text{ is an edge in } T \\ \text{directed towards the root } \Delta}} L_y^x \quad ^7$$

where V is the potential of the process X ⁸.

Proof. Suppose $|S| = n$. Choose an arbitrary order on S : $S = \{v_1, \dots, v_n\}$ and use Wilson's algorithm to construct a random spanning tree \mathcal{T} rooted at Δ . Set $v_0 = \Delta$. For a rooted spanning tree T , let $A_m(T)$ be the set of vertices in $T_{\{v_0, \dots, v_m\}}$ ⁹ for $m = 1, \dots, n$. Set $B_0(T) = \emptyset$. For $m = 1, \dots, n$, set $B_m(T) = \emptyset$ if v_m belongs $A_{m-1}(T)$. Otherwise, let $B_m(T)$ be the unique path joining v_m to $A_{m-1}(T)$ in T . We will calculate the conditional distribution of $B_m(\mathcal{T})$ given $A_{m-1}(\mathcal{T})$ for $m \geq 1$. Suppose that $v_m \notin A_{m-1}$. Let $(Y_t, t \geq 0)$ be the process $(X_t, t \geq 0)$ killed at the first jumping time after the process reaches the A_{m-1} . Then, Y is a transient Markov with generator

$$(L_Y)_y^x = \begin{cases} L_y^x & \text{for } x \text{ not contained in } \mathcal{T}_{\{v_0, \dots, v_{m-1}\}} \\ \delta_y^x L_x^x & \text{otherwise} \end{cases}$$

and potential V_Y such that

- $V_Y|_{A_{m-1}^c \times A_{m-1}^c} = V^{A_{m-1}^c}$;
- $(V_Y)_y^x = \sum_{z \in A_{m-1}^c} (V^{A_{m-1}^c})_z^x L_y^z$ for $x \in A_{m-1}^c, y \in A_{m-1}$;

⁷Recall that $L_\delta^x = - \sum_{y \in S} L_y^x$ for $x \in S$.

⁸Wilson's algorithm use the embedded Markov chain of X .

⁹Here, $T_{\{v_0, \dots, v_m\}}$ is the smallest sub-tree of T containing the same root with the set of vertices v_0, \dots, v_m .

- $V_Y|_{A_{m-1} \times A_{m-1}^c} = 0$;
- $(V_Y)_y^x = \delta_y^x \frac{1}{-L_x^x}$ for $x, y \in A_{m-1}$.

Let ∂_Y stand for the cemetery point of Y . Then conditionally on $\mathcal{T}_{v_0, \dots, v_{m-1}}$, the probability $B_m = ((z_0, z_1), (z_1, z_2), \dots, (z_p, z_{p+1}))$ with $z_0 = v_m, z_{p+1} \in A_{m-1}$ and $z_0, \dots, z_p \in A_{m-1}^c$ equals the probability that the loop-erased path obtained by Y is $(z_0, z_1, \dots, z_p, z_{p+1}, \partial_Y)$. According to Proposition 5.1, that conditional probability equals

$$\det((V_Y)_{A_m \setminus A_{m-1}}) \prod_{(x,y) \text{ is contained in } B_m} L_y^x = \det((V^{A_{m-1}^c})_{A_m \setminus A_{m-1}}) \prod_{(x,y) \text{ is contained in } B_m} L_y^x.$$

By Jacobi's formula,

$$\det(-L|_{A_{m-1}^c \times A_{m-1}^c}) \det((V^{A_{m-1}^c})_{A_m \setminus A_{m-1}}) = \det(-L|_{A_m^c \times A_m^c}).$$

Accordingly,

$$\det((V^{A_{m-1}^c})_{A_m \setminus A_{m-1}}) = \frac{\det(V^{A_{m-1}^c})}{\det(V^{A_m^c})}.$$

Therefore, if $v_m \notin A_{m-1}$, i.e. $A_{m-1} \neq A_m$,

$$\mathbb{P}[B_m = ((z_0, z_1), \dots, (z_p, z_{p+1})) | A_{m-1}] = \frac{\det(V^{A_{m-1}^c})}{\det(V^{A_m^c})} \prod_{(x,y) \text{ is contained in } B_m} L_y^x.$$

Trivially, if $v_m \in A_{m-1}$,

$$\mathbb{P}[B_m = \phi | A_{m-1}] = 1 = \frac{\det(V^{A_{m-1}^c})}{\det(V^{A_m^c})}$$

Finally, by multiplying all the conditional probability above, we find that

$$\mu_{ST, \Delta}(\mathcal{T} = T) = \det(V) 1_{\{T \text{ is a spanning tree rooted at } \Delta\}} \prod_{\substack{(x,y) \text{ is an edge in } T \\ \text{directed towards the root } \Delta}} L_y^x.$$

□

Remark 15. In Wilson's algorithm, the spanning tree is constructed by progressively adding new branches. For $k \in \mathbb{N}$, conditionally on the tree T_k that has been constructed at step k , the law of the new branch η_{k+1} to be added is associated with the Markov process X stopped at the next jump after reaching T_k . At the same time, we remove $\#T_{k+1} - \#T_k$ loops based on each vertex of η_{k+1} . If we partition these loops according to some independent Poisson-Dirichlet (0,1) distribution as in Proposition 5.4 and Remark 14 and we get an ensemble of loops $\mathcal{O}_{\eta_{k+1}}$. Conditionally on \mathcal{T} , \mathcal{O}_k is equal in law to $\mathcal{L}_1^{\{T_k\}^c} \setminus \mathcal{L}_1^{\{T_{k+1}\}^c}$. (By $\mathcal{L}_1^{\{T_k\}^c}$, we mean those loops in \mathcal{L}_1 that avoid the vertices of the tree T_k .) These $(\mathcal{O}_{\eta_k}, k \geq 1)$ are independent and it is the same for $(\mathcal{L}_1^{\{T_k\}^c} \setminus \mathcal{L}_1^{\{T_{k+1}\}^c}, k \geq 1)$.

It implies that $\bigcup_{k \geq 1} \mathcal{O}_{\eta_k}$ has the same law as $\mathcal{L}_1 = \bigcup_{k \geq 1} \mathcal{L}_1^{\{T_{k-1}\}^c} \setminus \mathcal{L}_1^{\{T_k\}^c}$. To summarize, in the Wilson's algorithm, we have removed $\#S$ loops based at each vertex of S . By partitioning all these loops according to independent Poisson-Dirichlet (0,1) distributions as in Proposition 5.4 and Remark 14, we recover the Poisson ensemble of loops \mathcal{L}_1 .

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